From Local SGD to Local Fixed-Point Methods for Federated Learning

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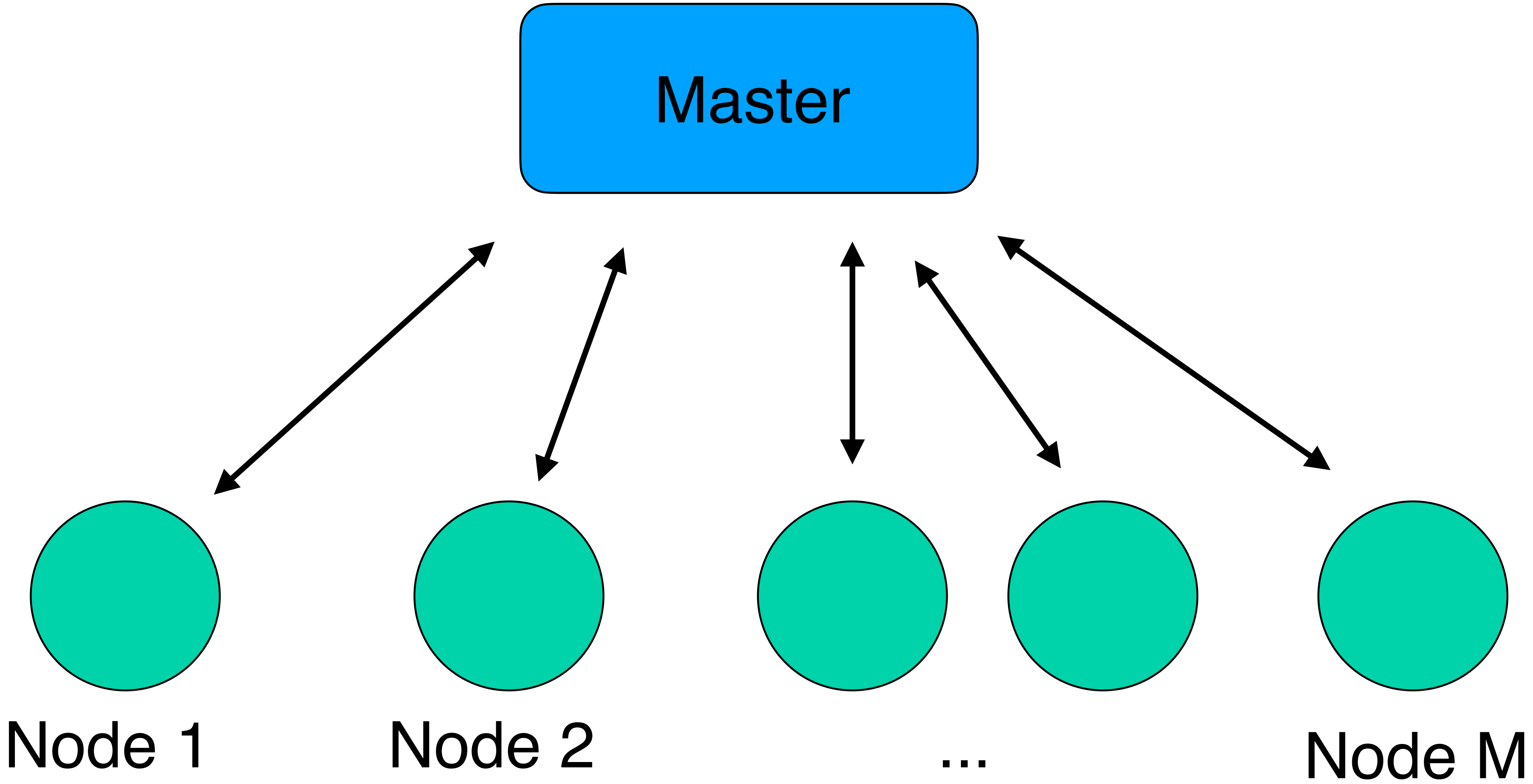
Distributed Algorithms



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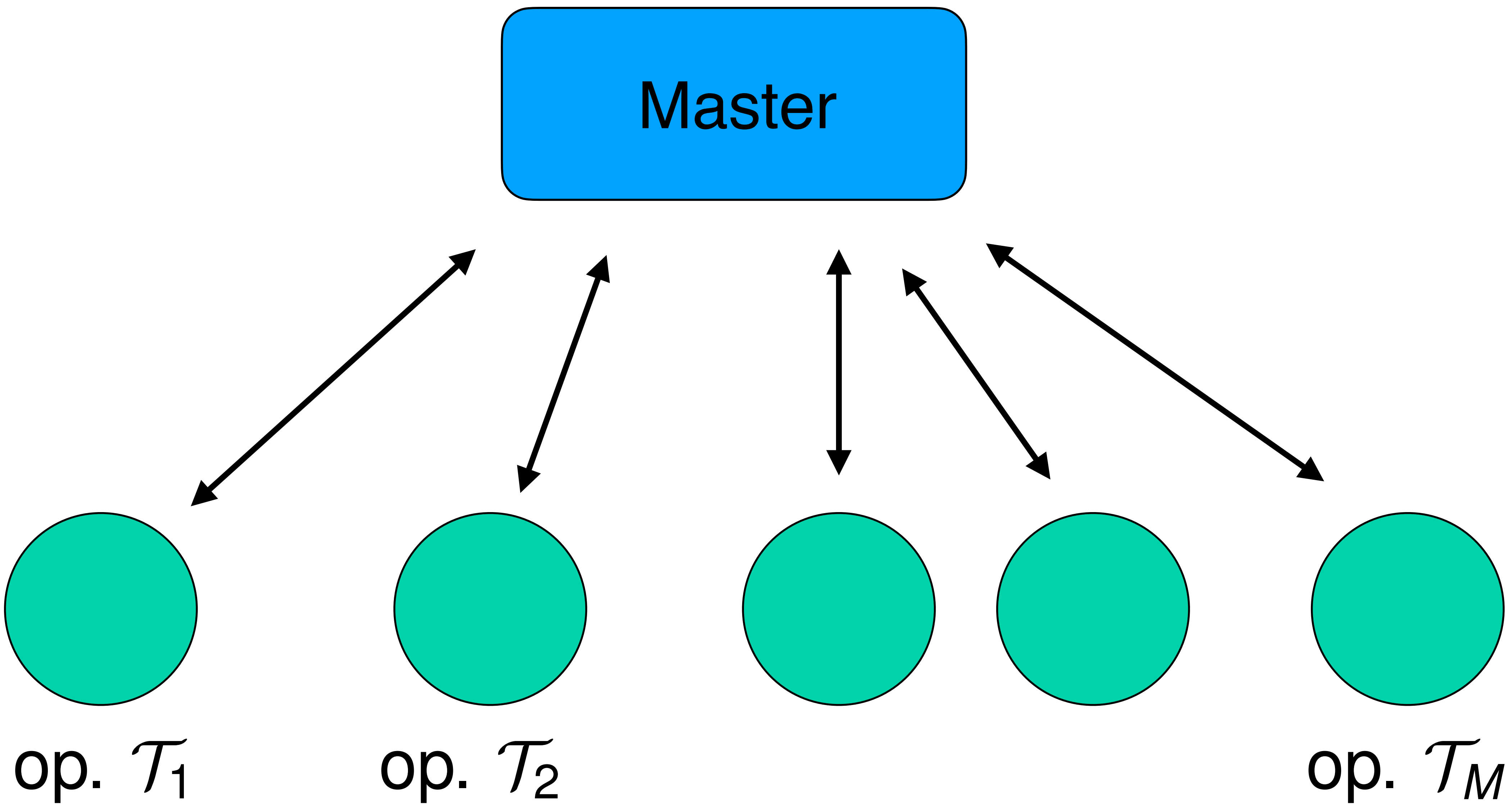

Distributed fixed-point problem

We define the average operator

$$\mathcal{T} : x \in \mathbb{R}^d \mapsto \frac{1}{M} \sum_{i=1}^{M} \mathcal{T}_i(x).$$

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A fixed-point algorithm iterates:

$$x^{k+1} = \mathcal{T}(x^k)$$

Optimization algorithms

Fixed-point algorithms:

* Find a minimizer of a function

Gradient descent:

$$x^{k+1} = x^k - \gamma \nabla F(x^k)$$

Proximal point algorithm:

$$x^{k+1} = \arg\min_x \; F(x) + \frac{1}{2\gamma}\|x - x^k\|^2$$

Optimization algorithms

Fixed-point algorithms:

* Find a minimizer of a function
    * Proximal splitting algorithms
    * Primal-dual algorithms
    * Cyclic or shuffled GD
    * (Block-)coordinate methods
    * Methods with inertia, momentum...
    * Conjugate gradient methods
    * Higher-order methods
    * ...

Fixed-point methods

Fixed-point algorithms:

* Find a minimizer of a function
* Find a saddle point of a convex-concave function
* Find a solution of a PDE
* Find an eigenvector
* Solve a monotone inclusion or variational inequality
* ...

Prior work: local gradient descent

* Stich, S. U. Local SGD Converges Fast and Communicates Little. In International Conference on Learning Representations, 2019.
* Khaled, A., Mishchenko, K., and Richtárik, P. First analysis of local GD on heterogeneous data. In *NeurIPS Workshop on Federated Learning for Data Privacy and Confidentiality*, 2019.
* Khaled, A., Mishchenko, K., and Richtárik, P. Tighter theory for local SGD on identical and heterogeneous data. In *The 23rd International Conference on Artificial Intelligence and Statistics (AISTATS 2020)*, 2020.
* Ma, C., Konecny, J., Jaggi, M., Smith, V., Jordan, M. I., Richtárik,P.,and Takác, M. Distributed optimization with arbitrary local solvers. *Optimization Methods and Software*, 32(4):813–848, 2017.
* Haddadpour, F. and Mahdavi, M. On the convergence of local descent methods in federated learning. *arXiv preprint arXiv:1910.14425*, 2019.

Algorithm 1

Algorithm 1 Local distributed fixed-point method

Input: Initial estimate $\hat{x}^0 \in \mathbb{R}^d$, stepsize $\lambda > 0$,
sequence of synchronization times $0 = t_0 < t_1 < \ldots$
Initialize: $x_i^0 := \hat{x}^0$, for $i = 1, \ldots, M$
for $k = 0, 1, \ldots$ **do**
  for $i = 1, 2, \ldots, M$ in parallel **do**
    $h_i^{k+1} := (1 - \lambda) x_i^k + \lambda \mathcal{T}_i(x_i^k)$
    if $k + 1 = t_n$, for some n, **then**
      Communicate h_i^{k+1} to master node
    else
      $x_i^{k+1} := h_i^{k+1}$
    end if
  end for
  if $k + 1 = t_n$, for some n, **then**
    At master node: $\hat{x}^{k+1} := \frac{1}{M} \sum_{i=1}^{M} h_i^{k+1}$
    Broadcast: $x_i^{k+1} := \hat{x}^{k+1}$ for all $i = 1, \ldots, M$
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end for

n-th epoch: sequence of iterations $k + 1 = t_{n-1} + 1, \dots, t_n$

Communication times

Nb of iterations in each epoch supposed bounded:

Assumption: $1 \leq t_n - t_{n-1} \leq H$, for every $n \geq 1$.

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Example:
$t_n = nH$

Analysis in the contractive case

- $t_n = nH$
- All $\mathcal{T}_i$ are χ-contractive, for $\chi \in [0, 1)$
  i.e. $\|\mathcal{T}_i(x) - \mathcal{T}_i(y)\| \leq \chi \|x - y\|, \;\; \forall x, y$

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- All $\mathcal{T}_i$ are χ-contractive, for $\chi \in [0, 1)$

We define the operator

$$\widetilde{\mathcal{T}} = \frac{1}{M}\sum_{i=1}^{M}\left(\lambda\mathcal{T}_i + (1-\lambda)\mathrm{Id}\right)^H$$

Then

$$\hat{x}^{(n+1)H} = \frac{1}{M}\sum_{i=1}^{M} h_i^{(n+1)H} = \widetilde{\mathcal{T}}(\hat{x}^{nH})$$

Analysis in the contractive case

Theorem 2.11 (linear convergence) The fixed point $x^\dagger$ of $\widetilde{\mathcal{T}}$ exists and is unique, and $\hat{x}^{nH}$ converges linearly to $x^\dagger$. More precisely,

(i) $\widetilde{\mathcal{T}}$ is ξ^H-contractive, with $\xi = \max\left(\lambda\chi + (1-\lambda), \lambda(1+\chi) - 1\right)$.

(ii) $\forall n \in \mathbb{N}, \quad \|\hat{x}^{(n+1)H} - x^\dagger\| \leq \xi^H \|\hat{x}^{nH} - x^\dagger\|$.

(iii) $\forall n \in \mathbb{N}, \quad \|\hat{x}^{nH} - x^\dagger\| \leq \xi^{nH} \|\hat{x}^0 - x^\dagger\|$.

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(iii) $\forall n \in \mathbb{N}, \quad \|\hat{x}^{nH} - x^\dagger\| \leq \xi^{nH} \|\hat{x}^0 - x^\dagger\|$.

Note: Without further knowledge, set $\lambda = 1$.

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Analysis in the contractive case

Theorem 2.14 (size of the neighborhood)
Suppose that $\lambda = 1$. So, $\xi = \chi$. Then

$$\|x^\dagger - x^\star\| \leq S,$$

where

$$S = \frac{\xi}{1-\xi}\frac{1-\xi^{H-1}}{1-\xi^{H}}\frac{1}{M}\sum_{i=1}^{M}\|\mathcal{T}_i(x^\star) - x^\star\|.$$

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Note 1: $S = 0$ if $H = 1$, or $M = 1$, or $\mathcal{T}_i = \mathcal{T}$, or $\xi = 0$.

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Note 1: $S = 0$ if $H = 1$, or $M = 1$, or $\mathcal{T}_i = \mathcal{T}$, or $\xi = 0$.

Note 2: If $H : 1 \nearrow +\infty$, $S : 0 \nearrow S^\infty$ with

$$S^\infty = \frac{\xi}{1-\xi}\frac{1}{M}\sum_{i=1}^{M}\|\mathcal{T}_i(x^\star) - x^\star\|.$$

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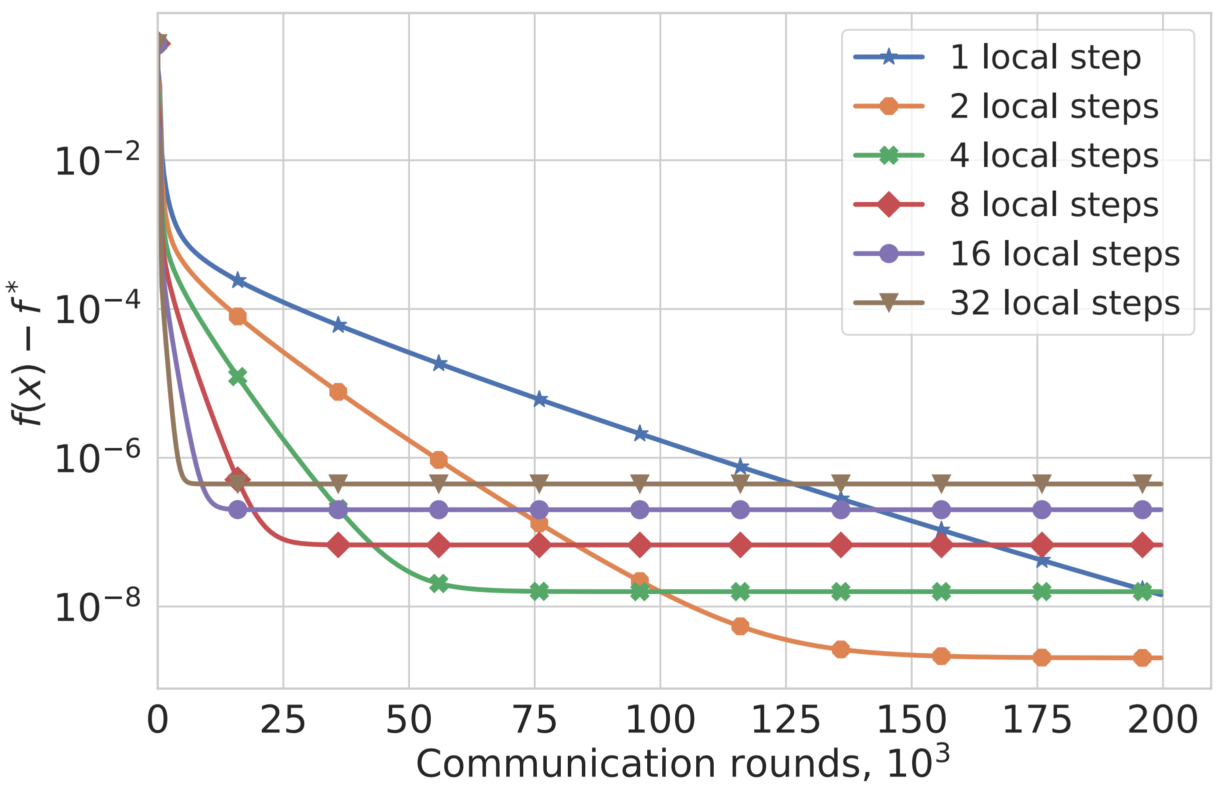
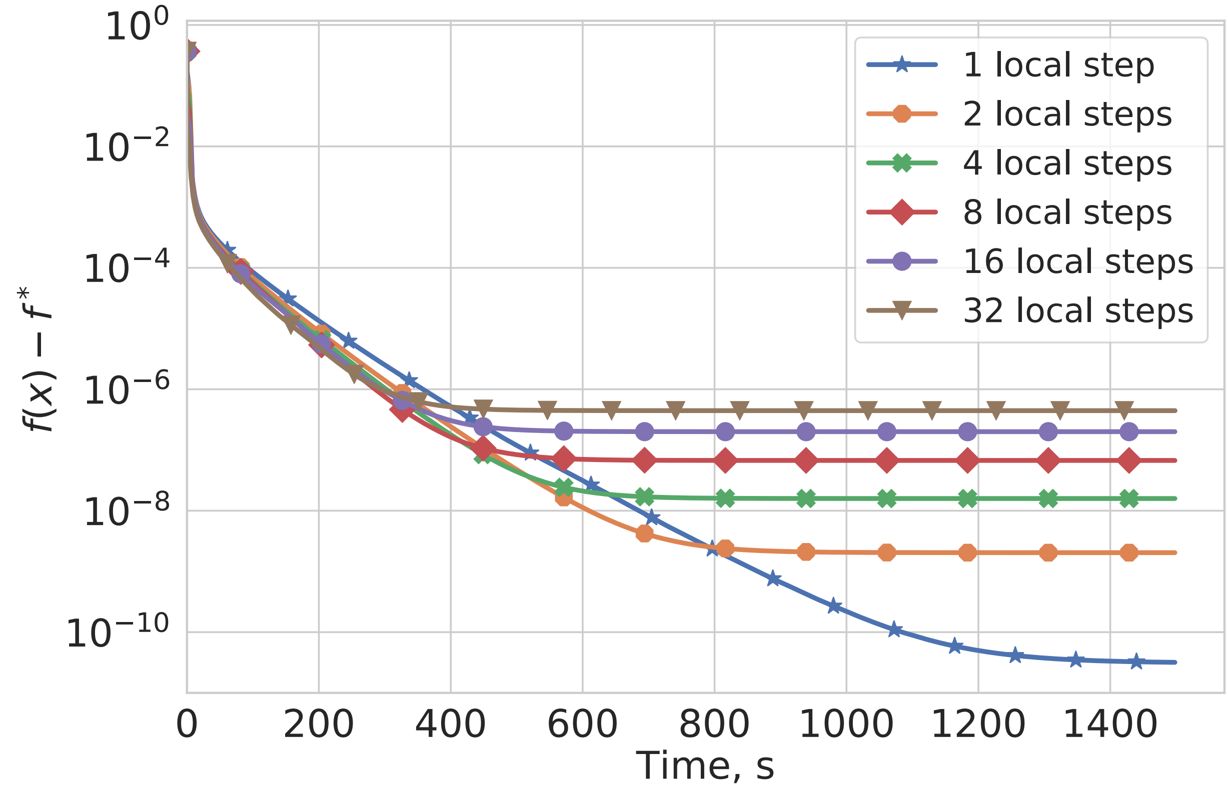
Corollary: For every $n \in \mathbb{N}$,

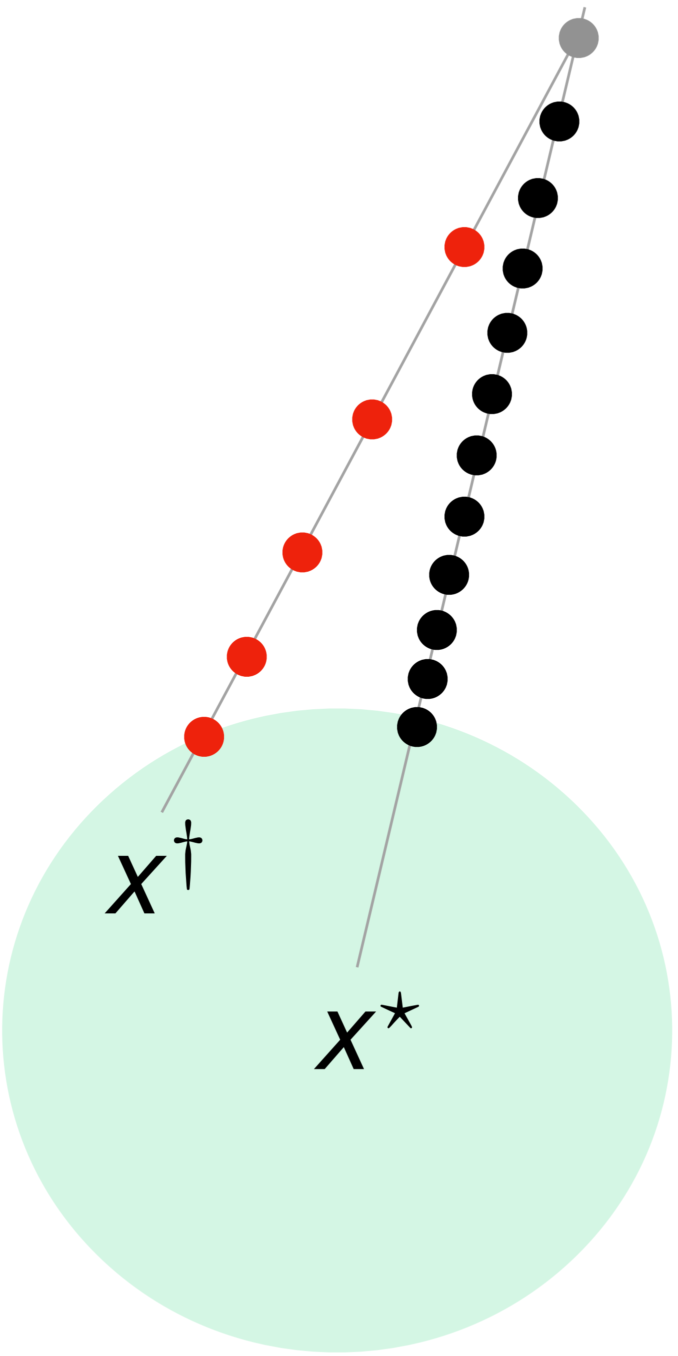
$$\begin{aligned}\|\hat{x}^{nH} - x^\star\| &\leq \xi^{nH}\|\hat{x}^0 - x^\dagger\| + S \\ &\leq \xi^{nH}(\|\hat{x}^0 - x^\star\| + S) + S.\end{aligned}$$

Results: logistic regression





Epsilon-accuracy



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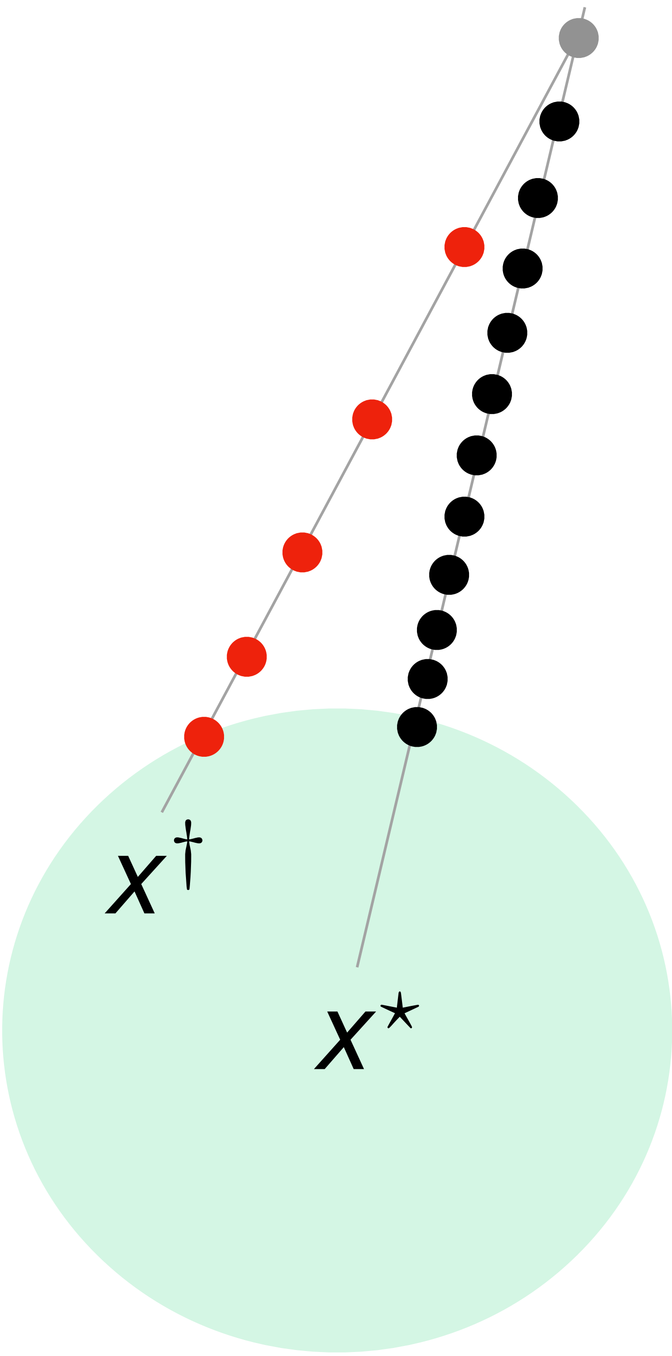

Note:

Local GD:

$O\left(\frac{L}{\mu}\frac{1}{H}\log\left(\frac{1}{\epsilon}\right)\right)$

but

$H = O(1+\epsilon)$

$O\left(\frac{L}{\mu}\log\left(\frac{1}{\epsilon}\right)\right)$

Analysis in the non-contractive case

- $t_n = nH$ ☞ convergence to $x^\dagger$, a fixed point of $\widetilde{\mathcal{T}} = \frac{1}{M}\sum_{i=1}^{M}\left(\lambda\mathcal{T}_i + (1-\lambda)\mathrm{Id}\right)^H$
- sublinear rates on $\|\hat{x}^{(n+1)H} - \hat{x}^{nH}\|^2$ or $\|\hat{x}^k - \mathcal{T}(\hat{x}^k)\|^2$
- $t_n = nH$ ☞ convergence w.r.t. nb. epochs 1 to H times faster

Algorithm 2

Algorithm 2 Randomized distributed fixed-point method

Input: Initial estimate $\hat{x}^0 \in \mathbb{R}^d$, stepsize $\lambda > 0$, communication probability $0 < p \leq 1$
Initialize: $x_i^0 = \hat{x}^0$, for all $i = 1, \dots, M$
for $k = 1, 2, \dots$ **do**
  for $i = 1, 2, \dots, M$ in parallel **do**
    $h_i^{k+1} \coloneqq (1 - \lambda) x_i^k + \lambda \mathcal{T}_i(x_i^k)$
  end for
  Flip a coin and
  with probability p **do**
    Communicate h_i^{k+1} to master, for $i = 1, \dots, M$
    At master node: $\hat{x}^{k+1} \coloneqq \frac{1}{M} \sum_{i=1}^{M} h_i^{k+1}$
    Broadcast: $x_i^{k+1} \coloneqq \hat{x}^{k+1}$, for all $i = 1, \dots, M$
  else, with probability $1 - p$, **do**
    $x_i^{k+1} \coloneqq h_i^{k+1}$, for all $i = 1, \dots, M$
end for

Analysis of Algorithm 2

Assumption 3.1

$(1+\rho)\|\mathcal{T}_i(x) - \mathcal{T}_i(y)\|^2 \leq \|x-y\|^2 - \|x - \mathcal{T}_i(x) - y + \mathcal{T}_i(y)\|^2$

for some $\rho > 0$

Lyapunov function:

$$\Psi^k := \|\hat{x}^k - x^\star\|^2 + \frac{5\lambda}{p}\frac{1}{M}\sum_{i=1}^{M}\left\|x_i^k - \hat{x}^k\right\|^2$$

Analysis of Algorithm 2

Assumption 3.1

$(1+\rho)\|\mathcal{T}_i(x)-\mathcal{T}_i(y)\|^2 \leq \|x-y\|^2 - \|x-\mathcal{T}_i(x)-y+\mathcal{T}_i(y)\|^2$
for some $\rho > 0$

Lyapunov function:

$$\Psi^k := \|\hat{x}^k - x^\star\|^2 + \frac{5\lambda}{p}\frac{1}{M}\sum_{i=1}^{M}\left\|x_i^k - \hat{x}^k\right\|^2$$

For λ small enough:

Theorem 3.2

$$\mathbb{E}[\Psi^k] \leq \left(1-\min\left(\frac{\lambda\rho}{1+\rho},\frac{p}{5}\right)\right)^k \Psi^0 + \frac{150}{\min\left(\frac{\lambda\rho}{1+\rho},\frac{p}{5}\right)p^2}\frac{\lambda^3}{M}\sum_{i=1}^{M}\|x^\star - \mathcal{T}_i(x^\star)\|^2$$

Conclusion

Local steps: good to achieve a medium-accuracy solution faster, if communication is the bottleneck