Analytic Marching: An Analytic Meshing Solution from Deep Implicit Surface Networks

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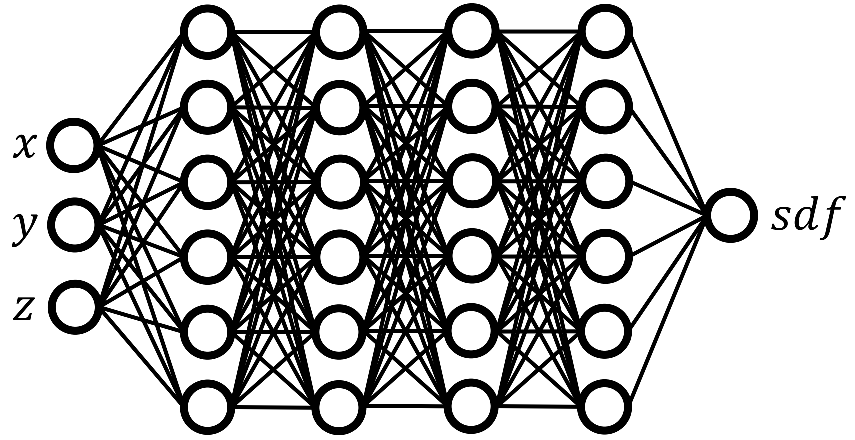
Introduction - Surface representation

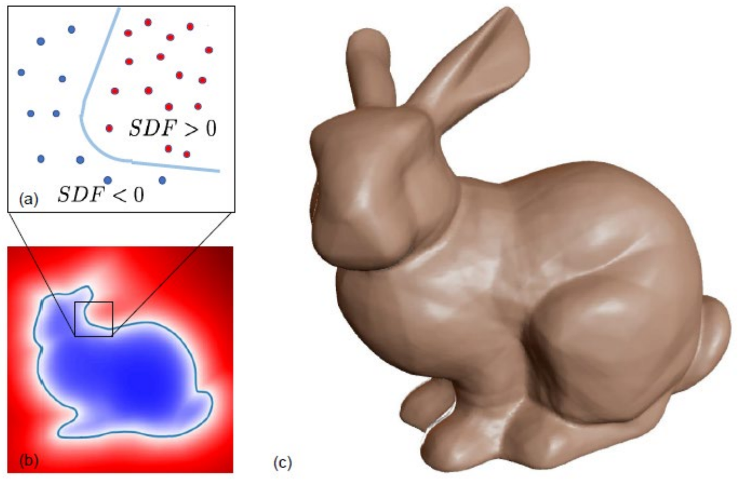
Explicit representation

- ✓ Polygonal mesh is piecewise linear approximation to the 3D surface.

Implicit representation

- ✓ The surface is represented as the zero-level isosurface of an implicit function.
- ✓ One of the well-known implicit functions is Signed Distance Field (SDF). Its magnitude represents the distance to the surface, and its sign indicates interior or exterior of that object.
- ✓ In deep learning, implicit function is usually implemented as a Multi-Layer Perceptron (MLP) with Rectified Linear Units (ReLUs).





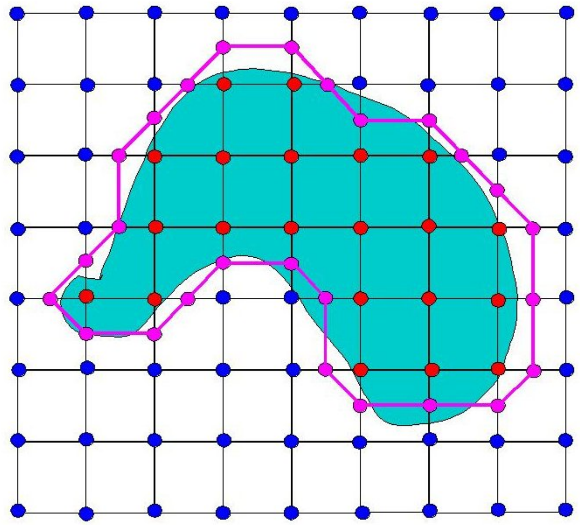
Introduction - Isosurface extraction

Isosurface extraction (meshing)

- ✓ **Definition:** The conversion from an implicit field to an explicit surface mesh.
- ✓ In deep mesh reconstruction, they typically take a final step of Marching Cubes (MC) to obtain the mesh reconstruction results.

$$\begin{tikzcd}\boxed{\textbf{Input}} \arrow[r] & \boxed{\textbf{Implicit function}} \arrow[r, "\textbf{Isosurface extraction}"] & \boxed{\textbf{Mesh}}\end{tikzcd}$$

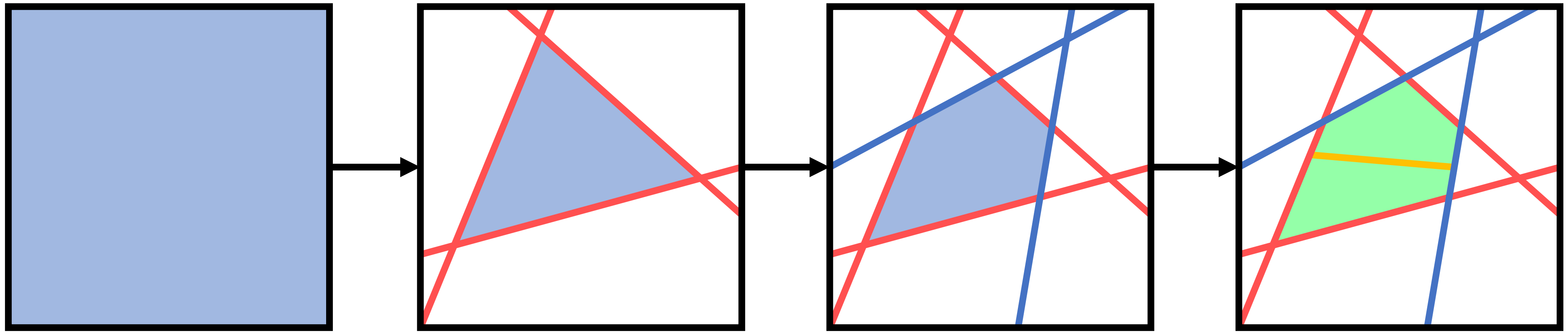
- ✓ While promising, they often suffer from loss of precision learned in the MLPs, due to the discretization nature of Marching Cubes.
- ✓ They require lots of points sampled from 3D space, which takes time, and the resulting mesh is just an approximation to the underlying isosurface.



Introduction - Local linearity

Piecewise linear property

✓ Observing that polygonal meshes have the same piecewise linear properties as MLPs, and motivated by the knowledge that a ReLU based MLP partitions its input space into a number of linear regions, we identify from these regions analytic cells and analytic faces that are associated with zero-level isosurface of the implicit function.



🟥	Hyperplane arrangement at the first layer	🟨	Analytic face
🟦	Hyperplane arrangement at the second layer	🟩	Analytic cell

Theory of analytic geometry of neural networks - Basis

Multi-Layer Perceptron (MLP)

An MLP of L hidden layers takes an input $x \in \mathbb{R}^{n_0}$, and layer-wisely computes $x_l = g(W_l x_{l-1})$, where g is the point-wise, ReLU based activation function. We compactly write the MLP as a mapping $Tx = g(W_L \dots g(W_1 x))$.

Functionals

Any k^{th} neuron, $k \in \{1, \dots, n_l\}$, of an l^{th} layer of the MLP T specifies a functional defined as

$$a_{lk}(\boldsymbol{x}) = \pi_k \boldsymbol{g}(\boldsymbol{W}_l \boldsymbol{g}(\boldsymbol{W}_{l-1} \dots \boldsymbol{g}(\boldsymbol{W}_1 \boldsymbol{x})))$$

where π_k denotes an operator that projects onto the k^{th} coordinate.

All the neurons at layer l define a functionals as

$$\boldsymbol{a}_l(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{W}_l \boldsymbol{g}(\boldsymbol{W}_{l-1} \dots \boldsymbol{g}(\boldsymbol{W}_1 \boldsymbol{x})))$$

Region/Cell

Hyperplanes specified by all the n_l neurons of layer l from a hyperplane arrangement, which partitions the feature space of layer $l-1$ into multiple linear regions.

Let $\mathcal{R}$ denote the set of all linear regions/cells in $\mathbb{R}^{n_0}$ that are possibly achieved by T.

Theory of analytic geometry of neural networks - Basis

State functionals

For a k^{th} neuron of an l^{th} layer of an MLP T, with $k \in \{1, \dots, n_l\}$ and $l \in \{1, \dots, L\}$, its state functional of neuron activation is defined as

$$s_{lk}(\boldsymbol{x}) = \begin{cases} 1 & \textit{if } a_{lk}(\boldsymbol{x}) > 0 \\ 0 & \textit{if } a_{lk}(\boldsymbol{x}) \leq 0 \end{cases}$$

which gives the state functional of layer l as

$$\boldsymbol{s}_l(\boldsymbol{x}) = [s_{l1}(\boldsymbol{x}), \dots, s_{ln_l}(\boldsymbol{x})]^\top$$

and the state functional of MLP T as

$$\boldsymbol{s}(\boldsymbol{x}) = [s_1(\boldsymbol{x})^\top, \dots, s_L(\boldsymbol{x})^\top]^\top$$

Since $s(x)$ is fixed for all x that fall in the same region $r \in \mathcal{R}$, we use $s(r) \in \{0,1\}^N$ to label this region, where N is the number of all hidden neurons.

Theory of analytic geometry of neural networks - Basis

Region-wise linear mappings

Given a ReLU based MLP T of L hidden layers, for any region/cell $r \in \mathcal{R}$, its associated linear mapping T^r is defined as

$$\boldsymbol{T}^r = \prod_{l=1}^{L} \boldsymbol{W}_l^r \qquad \boldsymbol{W}_l^r = diag(\boldsymbol{s}_l(r))\boldsymbol{W}_l$$

Functional of SDF

We stack on top of T a regression function $f\colon \mathbb{R}^{n_L} \to \mathbb{R}$, giving rise to a functional of SDF as

$$F(\boldsymbol{x}) = f \circ \boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{w}_f^\top \boldsymbol{g}(\boldsymbol{W}_L \ldots \boldsymbol{g}(\boldsymbol{W}_1\boldsymbol{x}))$$

Neuron-wise linear mappings

With above definitions, we can give a linear mapping of a k^{th} neuron of layer l from input space, and the linear mapping of output neuron.

$$\boldsymbol{a}_{lk}^r = \pi_k \prod_{i=1}^{l} \boldsymbol{W}_i^r \qquad \boldsymbol{a}_F^r = \boldsymbol{w}_f^\top \boldsymbol{T}^r = \boldsymbol{w}_f^\top \prod_{i=1}^{L} \boldsymbol{W}_i^r$$

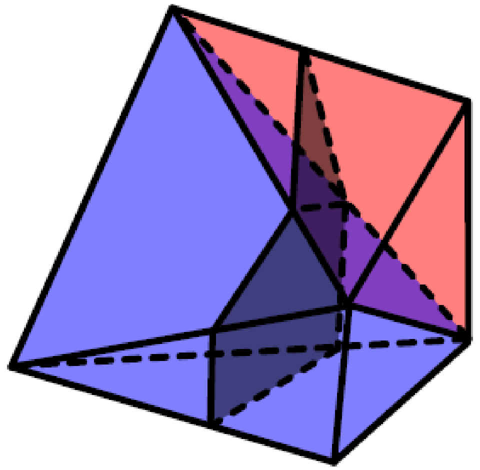
Theory of analytic geometry of neural networks - Analytic cells / faces

Feasible region

Let $\{\tilde{r} \in \tilde{\mathcal{R}}\}$ denote the subset of regions in $\mathcal{R}$ that have the intersection property.

For any $x \in \tilde{r}$, it must satisfy the following system of inequalities.

$$(\boldsymbol{I} - 2\mathrm{diag}(\boldsymbol{s}(\tilde{r}))\boldsymbol{A}^{\tilde{r}}\boldsymbol{x} = \begin{bmatrix} (1 - 2s_{11}(\tilde{r}))\boldsymbol{a}_{11}^{\tilde{r}} \\ \vdots \\ (1 - 2s_{lk}(\tilde{r}))\boldsymbol{a}_{lk}^{\tilde{r}} \\ \vdots \\ (1 - 2s_{Ln_L}(\tilde{r}))\boldsymbol{a}_{Ln_L}^{\tilde{r}} \end{bmatrix} \boldsymbol{x} \preceq 0$$



Analytic cell (with zero-level isosurface)

When the region is bounded, the above inequalities essentially forms a polyhedral cell defined as

$$\mathcal{C}_F^{\tilde{r}} = \{\boldsymbol{x} \in \mathbb{R}^3 | (\boldsymbol{I} - 2\mathrm{diag}(\boldsymbol{s}(\tilde{r}))\boldsymbol{A}^{\tilde{r}}\boldsymbol{x} \preceq 0\}$$

Analytic face

We define the polygonal face that is an intersection of analytic cell $\tilde{r}$ and surface $\mathcal{Z}$.

$$\mathcal{P}_F^{\tilde{r}} = \{\boldsymbol{x} \in \mathbb{R}^3 | \boldsymbol{a}_F^{\tilde{r}}\boldsymbol{x} = 0, (\boldsymbol{I} - 2\mathrm{diag}(\boldsymbol{s}(\tilde{r}))\boldsymbol{A}^{\tilde{r}}\boldsymbol{x} \preceq 0\}$$

Theory of analytic geometry of neural networks - Analytic closure theorem

Analytic closure theorem

We have the following theorem that characterizes the conditions under which analytic faces guarantee to connect and form a closed, piecewise planar surface.

Assume that the zero-level isosurface $\mathcal{Z}$ of a SDF $F = f \circ \boldsymbol{T}$ defines a closed, piecewise planar surface. If for any region/cell $r \in \mathcal{R}(\boldsymbol{T})$, its associated linear mapping $\boldsymbol{T}^r$ and the induced plane $\boldsymbol{a}_F^r = \boldsymbol{w}_f^\top \boldsymbol{T}^r$ are uniquely defined, i.e. $\boldsymbol{T}^r \neq \beta \boldsymbol{T}^{r'}$ and $\boldsymbol{a}_F^r \neq \beta \boldsymbol{a}_F^{r'}$ for any region pair of r and r', where β is an arbitrary scaling factor, then analytic faces $\{\mathcal{P}_F^{\tilde{r}}\}$ connect and exactly form the surface $\mathcal{Z}$.

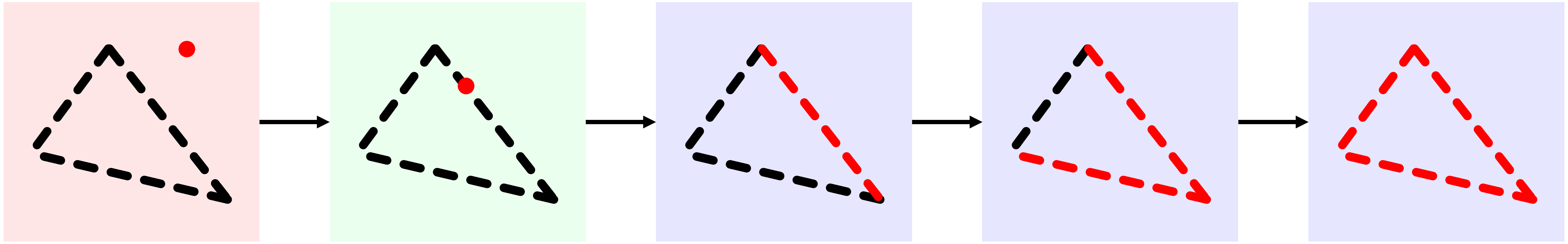
Note that the conditions assumed in this theorem are practically reasonable up to a numerical precision of the learned weights in the SDF $F = f \circ \boldsymbol{T}$.

The proof of the theorem also suggests an algorithm to identify the polygonal faces of a surface mesh learned by F, which is to be presented shortly.

Analytic Marching - Overview

Motivation

Given a learned SDF $F = f \circ T$ whose zero-level isosurface defines a closed, piecewise planar surface, a theorem proved by us suggests that obtaining the mesh concerns with identification of analytic faces $\mathcal{P}_F^{\tilde{r}}$ in analytic cell $\{\mathcal{C}_F^{\tilde{r}} | \tilde{r} \in \tilde{\mathcal{R}}\}$. To this end, we propose an algorithm of Analytic Marching that marches among analytic cells $\{\mathcal{C}_F^{\tilde{r}} | \tilde{r} \in \tilde{\mathcal{R}}\}$ to identify vertices and edges of the polygonal faces $\{\mathcal{P}_F^{\tilde{r}} | \tilde{r} \in \tilde{\mathcal{R}}\}$.



① Randomly initialize one point in 3D space
② Move the point to the surface
③ Identify one analytic face
④ Infer new states of neighbor faces
⑤ Repeatedly identify analytic faces

Analytic Marching - Details

The steps of Analytic Marching are as follows:

1. Identify at least one point $\boldsymbol{x} \in \mathcal{Z}$.
   Given the parametric model F and an arbitrarily initialized point $x \in \mathbb{R}^3$, this can be simply achieved by solving the following problem with stochastic gradient descent (SGD).

$$\min_{\boldsymbol{x} \in \mathbb{R}^3} |F(\boldsymbol{x})|$$

2. Compute state vector $s(x)$, and initialize an active set $\mathcal{S}^\bullet = \emptyset$ and an inactive set $\mathcal{S}^\circ = \emptyset$, push $s(x)$ into $\mathcal{S}^\bullet$.
3. Take an active state s_i from $\mathcal{S}^\bullet$, which specifies its analytic cell $\mathcal{C}_F^{\tilde{r}_i}$ and analytic face $\mathcal{P}_F^{\tilde{r}_i}$.
4. Enumerate all the vertices of the analytic face defined by

$$\mathcal{P}_F^{\tilde{r}_i} = \{\boldsymbol{x} \in \mathbb{R}^3 | \boldsymbol{a}_F^{\tilde{r}_i} \boldsymbol{x} = 0, (\boldsymbol{I} - 2\mathrm{diag}(\boldsymbol{s}(\tilde{r}_i))\boldsymbol{A}^{\tilde{r}_i}\boldsymbol{x} \preceq 0\}$$

   Note that it can be done simply by brute-force vertices enumeration.
   Record all the boundary planes $\{\widehat{H}_{lk}^{\tilde{r}_i}\}$ of $\mathcal{C}_F^{\tilde{r}_i}$ that give valid vertices.
5. Push s_i out of the active set $\mathcal{S}^\bullet$, and into the inactive set $\mathcal{S}^\circ$.
6. Infer the state vector $\widehat{s}_i$ of adjacent faces, and push $\{\widehat{s}_i | \widehat{s}_i \notin \mathcal{S}^\circ\}$ into the active set $\mathcal{S}^\bullet$.
   Note that the analytic cell connecting $\mathcal{C}_F^{\tilde{r}_i}$ at a boundary $\widehat{H}_{lk}^{\tilde{r}_i}$ has its state vector switching only at the k^{th} neuron of layer l.
7. Repeat steps 3 to 6, until the active set $\mathcal{S}^\bullet$ is cleared up.

Analytic Marching - Analysis

Computational complexities

Assume that the SDF $F = f \circ T$ is built on an MLP of L hidden layers, each of which has n neurons, and thus $N = nL$

The complexity of our Analytic Marching algorithm has an order of $\mathcal{O}\left((n/2)^{2(L-1)}|\mathcal{V}_{\mathcal{P}}|n^4L^2\right)$

Ideally, its complexity is exponential w.r.t. depth L and polynomial w.r.t. width n

Practical implementations

The proposed analytic marching can be naturally implemented in parallel, and initialization of multiple surface points would help recover components of the surface that are possibly isolated

Training objective

For any $x \in \mathbb{R}^3$, let $d(x)$ denote its ground-truth value of signed distance to the surface. We use the following regularized objective to train a SDF $F = f \circ T$,

$$\min_{F=f\circ\boldsymbol{T}} \mathbb{E}_{\boldsymbol{x}\sim\mathbb{R}^3}\left|F(\boldsymbol{x})-d(\boldsymbol{x})\right|+\alpha\left|\|\partial F(\boldsymbol{x})/\partial\boldsymbol{x}\|-1\right|$$

The unit gradient regularizer aims to promote learning of a smooth gradient field

Experiments

Dataset

ShapeNet - A repository of many 3D shapes

Evaluation metrics

Chamfer Distance (CD), Earth Movers Distance (EMD),

Intersection over Union (IoU), F-score (F),

Wall-clock time (Time), Number of faces (Face No.).

Ablation study – Depth and Width

Architecture	CD($\times 10^{-1}$)	EMD($\times 10^{-3}$)	IoU(%)	F@τ(%)	F@2τ(%)	Face No.	Time(sec.)
D4-W90	6.16	8.60	86.5	84.8	95.4	97865	17.0
D6-W60	5.29	8.00	86.9	85.5	95.8	100179	15.0
D8-W45	5.67	8.12	85.7	84.2	95.4	93482	9.75
D10-W90	3.64	5.85	90.2	88.1	98.6	421840	173
D15-W60	3.85	6.23	88.9	87.4	97.5	336044	86.8
D20-W45	4.21	6.89	86.1	86.6	95.5	253970	47.3

“D” stands for depth, “W” stands for width.

Given a fixed number of neurons, it seems that properly deep networks are advantageous in terms of precision-efficiency trade off

Experiments

Results of other object categories

It tells that different categories have different complexity. It seems that the complexities of "Chair" are higher, and those of "Airplane" are lower.

Category	CD($\times10^{-1}$)	EMD($\times10^{-3}$)	IoU(%)	F@τ(%)	F@2τ(%)	Face No.	Time(sec.)
Rifle	5.29	8.00	86.9	85.5	95.8	100179	15.0
Chair	7.27	8.01	89.6	52.9	96.1	182624	25.3
Airplane	3.25	5.32	89.4	89.8	97.9	131074	19.0
Sofa	6.78	6.65	96.6	53.2	97.9	156863	22.0
Table	6.98	7.16	90.7	51.6	96.8	163395	22.7
Mean	5.91	7.03	90.6	66.6	96.9	146827	20.8

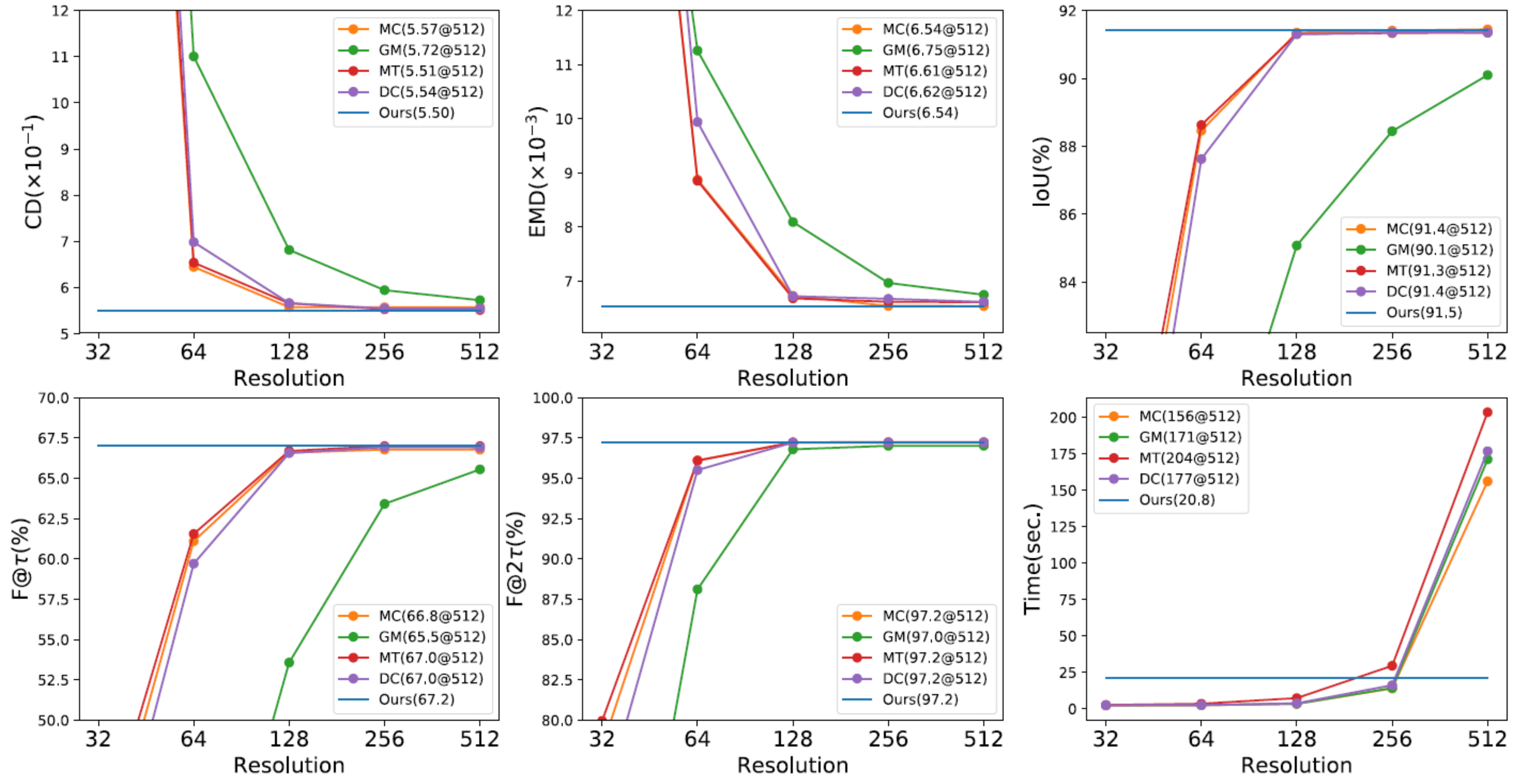
Comparisons with existing meshing algorithms

We compare our proposed algorithm (Analytic Marching, AM) with existing algorithms of Greedy Meshing (GM), Marching Cubes (MC), Marching Tetrahedra (MT), and Dual Contouring (DC).

Their meshing accuracy depends on the sampling resolution. We implement them under a range of sampling resolutions from 32^3 to a GPU memory limit of 512^3

Experiments

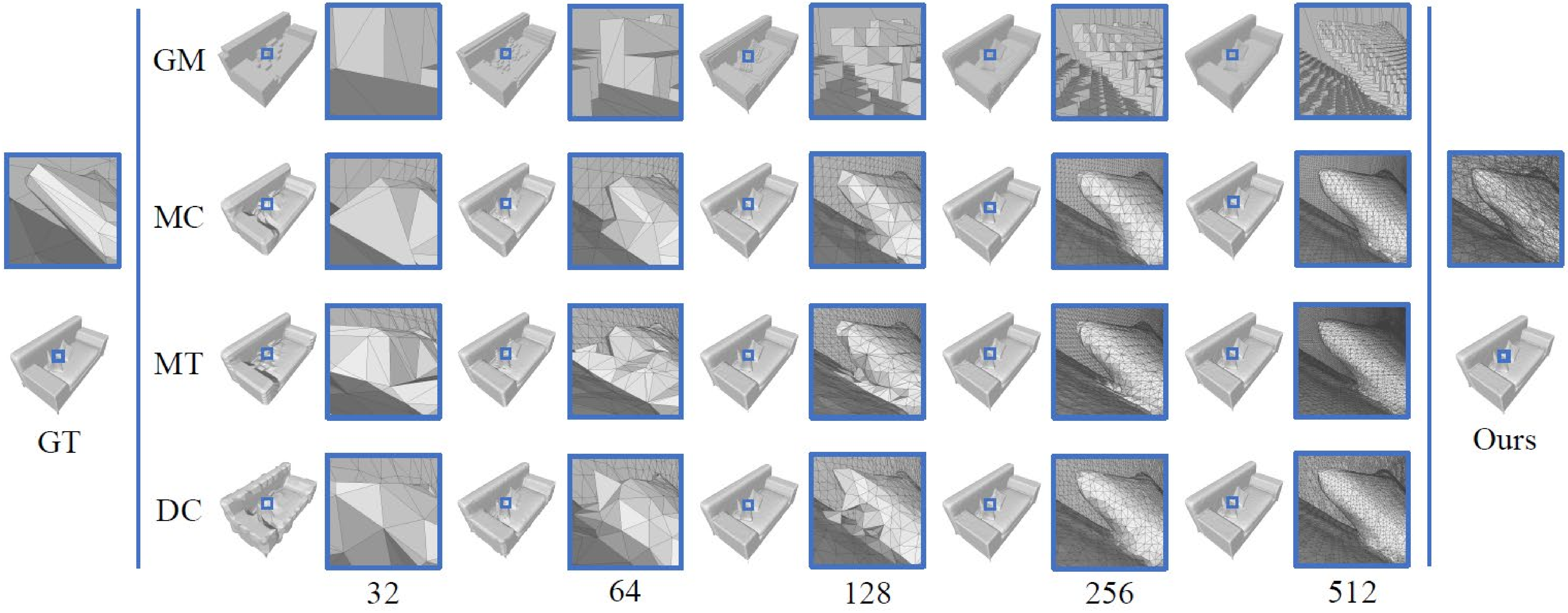
Comparisons with existing meshing algorithms



Under different evaluation metrics, recovery accuracies of these methods are upper bounded by our proposed one. Note that the dominating computations of competitors are implemented on GPU, which gives them unfair advantage of computational efficiency.

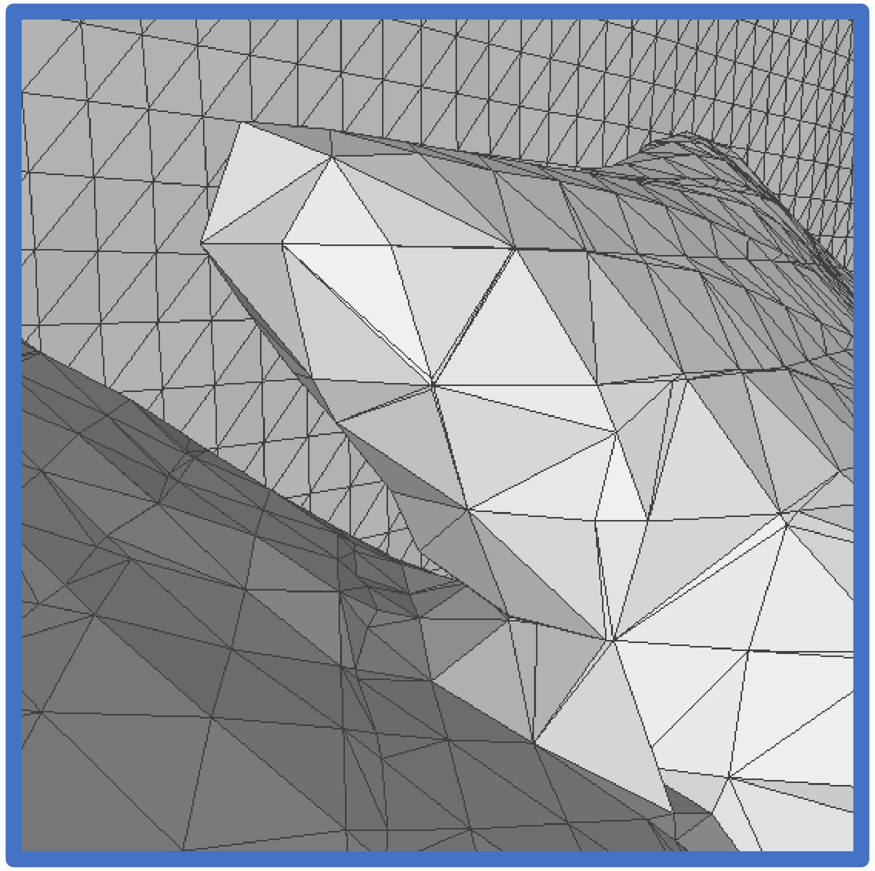
Experiments

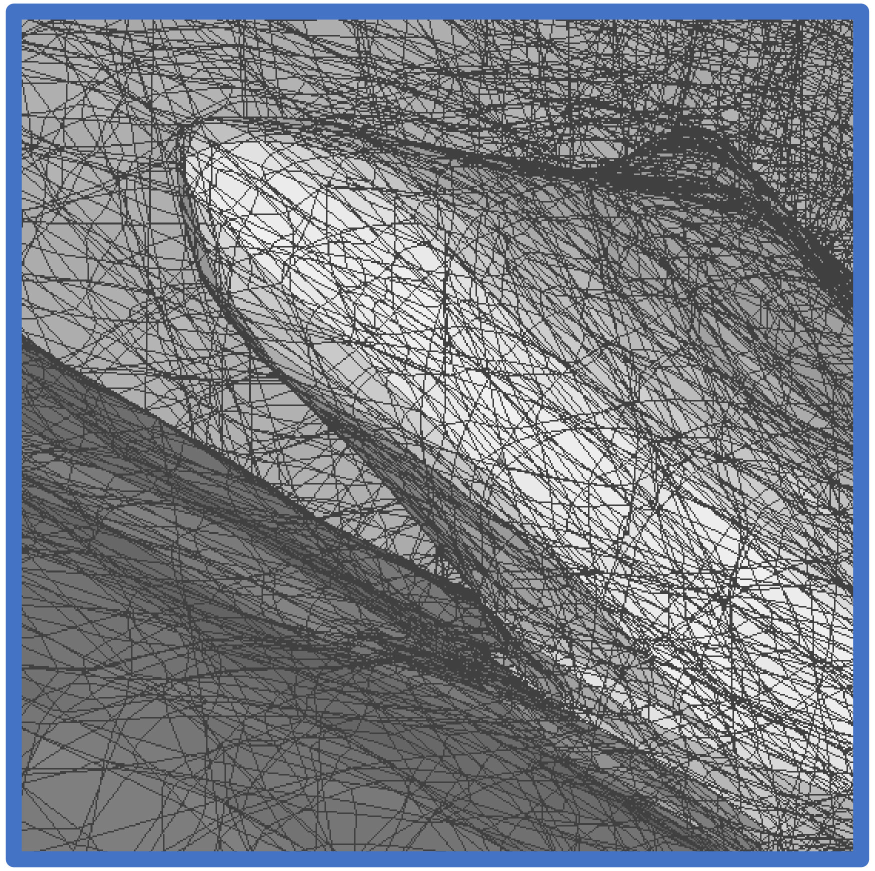
Comparisons with existing meshing algorithms



Experiments

Comparisons with existing meshing algorithms

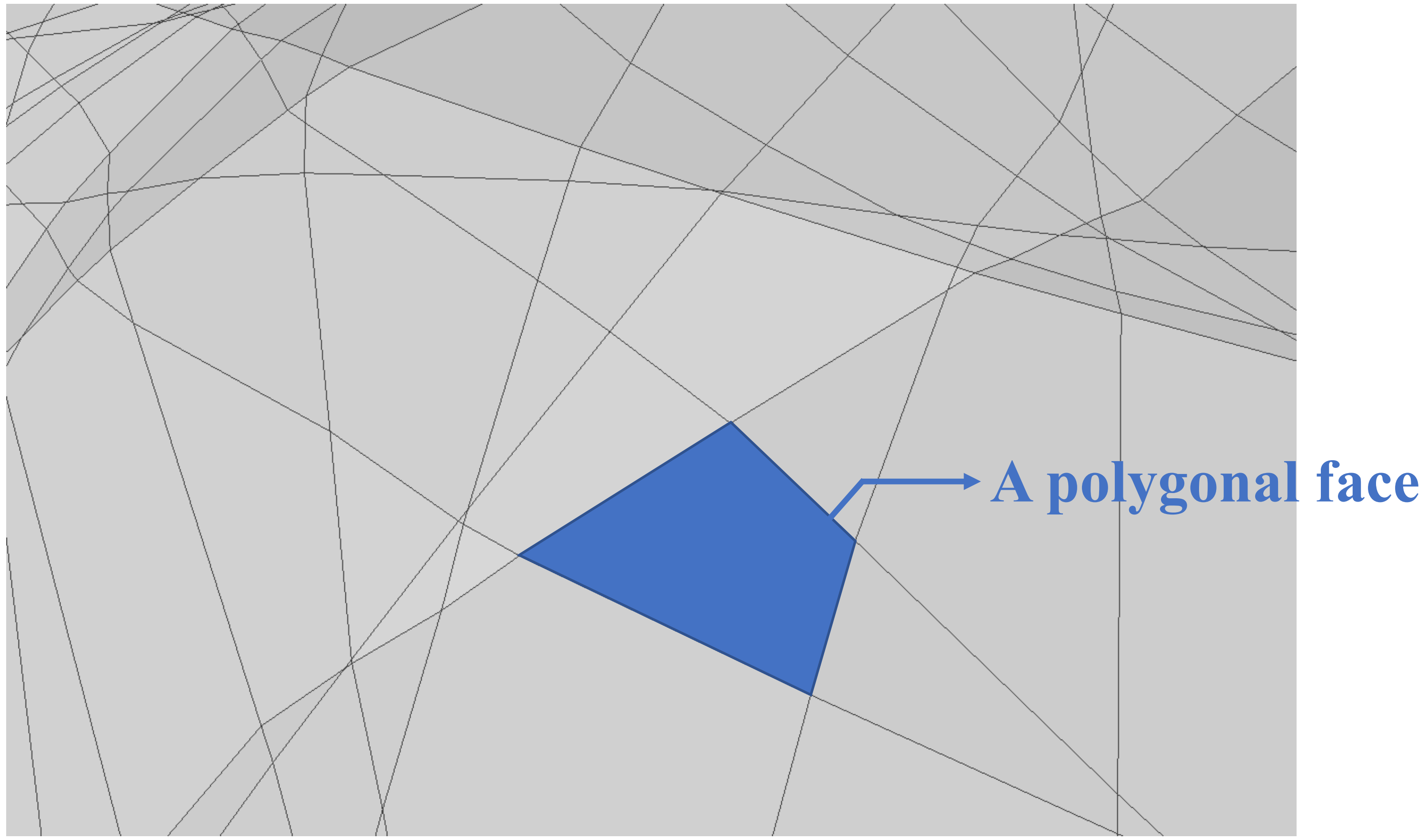

MC128


AM

Experiments

Local surface

The 3D mesh obtained by ours can be mathematically guaranteed to be a polygonal mesh



Thank you!