# Average-case Acceleration Through Spectral Density Estimation

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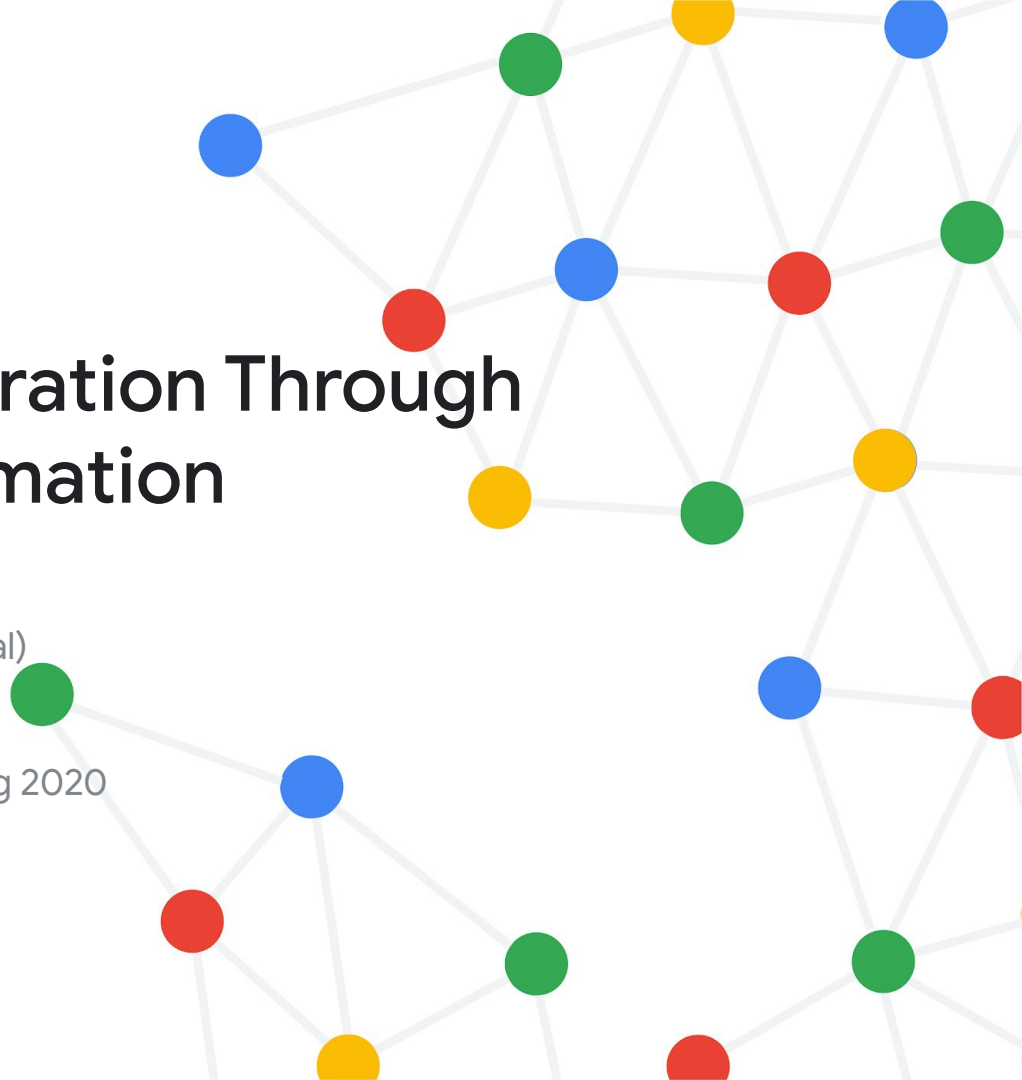
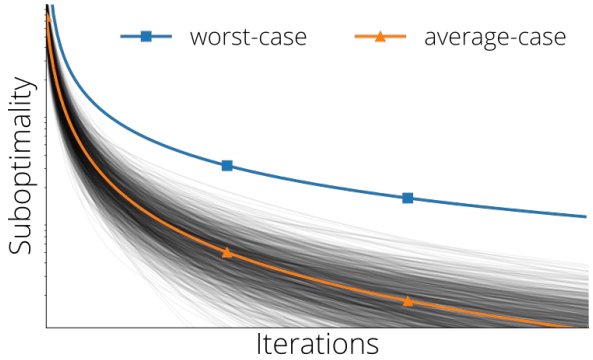


# Complexity Analysis in Optimization

Worst-case analysis

- ✓ Bound on the complexity for *any* input.
- ✗ Potentially worse than observed runtime.




Simplex method (Dantzig, '98, Spielman & Teng '04)

- ✗ Exponential worst-case.
- ✓ Runtime typically polynomial.

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# Average-case Complexity

- ✓ Complexity *averaged* over all problem instances.
- ✓ Representative of the typical complexity.

Better bounds, sometimes better algorithms

→ Quicksort (Hoare ’62): Fast average-case sorting

*Rarely used in optimization*

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## Main contributions

Average-case analysis for optimization on quadratics.

Optimal methods under this analysis.

# Problem Distribution: Random Quadratics

$$\min_{x \in \mathbb{R}^d} \left\{ f(x) \overset{\text{def}}{=} \frac{1}{2}(x - x^\star)^\top H (x - x^\star) \right\},$$

where $H$, $x^\star$ are random matrix, vector.

- ✓ exact runtime known depends on eigenvalues(**H**).
- ✓ shares (some) dynamics of real problems, e.g., Neural Tangent Kernel (Jacot et al., 2018).

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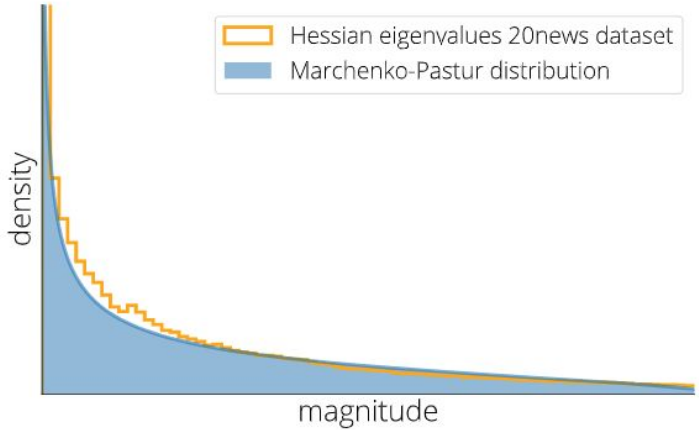
# Example: Random Least Squares

$$\min_{\boldsymbol{x}\in\mathbb{R}^d} \|\boldsymbol{A}\boldsymbol{x}-\boldsymbol{b}\|^2, \text{ with } \boldsymbol{b}=\boldsymbol{A}\boldsymbol{x}^\star$$

When elements of **A** are iid, standardized:

Spectrum of **H** will be close to Marchenko-Pastur.

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# Expected Error For Gradient-Based Methods

$$\text{expected error} = \mathbb{E}\|\boldsymbol{x}_t - \boldsymbol{x}^\star\|^2 = \overbrace{R^2}^{\text{initialization}} \int_{\mathbb{R}} \overbrace{P_t^2}^{\text{algorithm}} \overbrace{\mathrm{d}\mu}^{\text{problem}}$$

$R^2$ is the distance to optimum at initialization $\mathbb{E}\|\boldsymbol{x}_0 - \boldsymbol{x}^\star\|^2$.

Fixed

Problem difficulty represented by expected density Hessian eigenvalue $\mathrm{d}\mu$

$P_t$ is a polynomial of degree t determined from the optimization algorithm.

Flexible: algorithm design

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# Average-case Optimal Method

**Goal**: Find method with minimal expected error = $R^2 \int_{\mathbb{R}} P_t^2 \, d\mu$ (initialization: $R^2$; algorithm: $P_t^2$; problem: $d\mu$)

**Algorithms ↔ Polynomials**

Find **polynomial** $P_t$ of degree t that minimizes expected error (with proper normalization).

**Solution:** Polynomial of degree t, orthogonal wrt to $\lambda d\mu(\lambda)$ .

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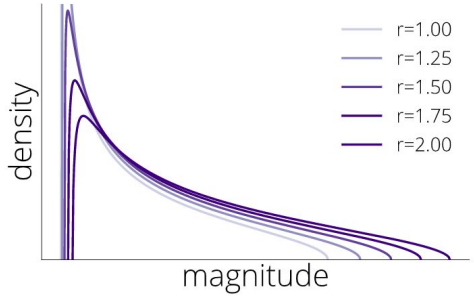
# Marchenko-Pastur Acceleration




**Model for** $d\mu$ = Marchenko-Pastur(r, $\sigma$).

r and $\sigma$ estimated from:

- Largest eigenvalue
- Trace of H

No need to know strong convexity constant.

**Algorithm**

$$\boldsymbol{x}_t = \boldsymbol{x}_{t-1} + \underbrace{a_t(r,\sigma)}_{\text{momentum}}(\boldsymbol{x}_{t-2} - \boldsymbol{x}_{t-1}) + \underbrace{b_t(r,\sigma)}_{\text{step-size}}\nabla f(\boldsymbol{x}_{t-1})$$

Simple momentum-like method, low memory requirements.

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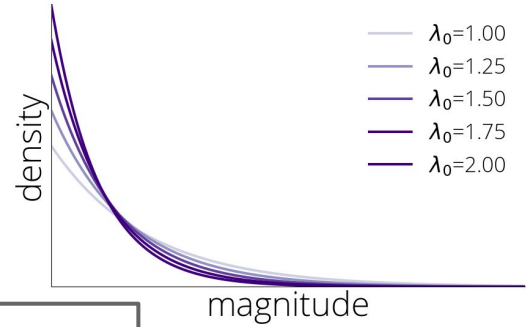
# Decaying Exponential Acceleration

**Model for** $d\mu$ = decaying exponential($\lambda_0$).

**Unbounded** largest eigenvalue. Only access to Tr(**H**).




**Algorithm**

$$\boldsymbol{x}_t = \boldsymbol{x}_{t-1} + \frac{t-1}{t+1}(\boldsymbol{x}_{t-1} - \boldsymbol{x}_{t-2}) - \frac{\lambda_0}{t+1}\nabla f(\boldsymbol{x}_{t-1})$$

- Decaying step-size
- Similar to Polyak averaging

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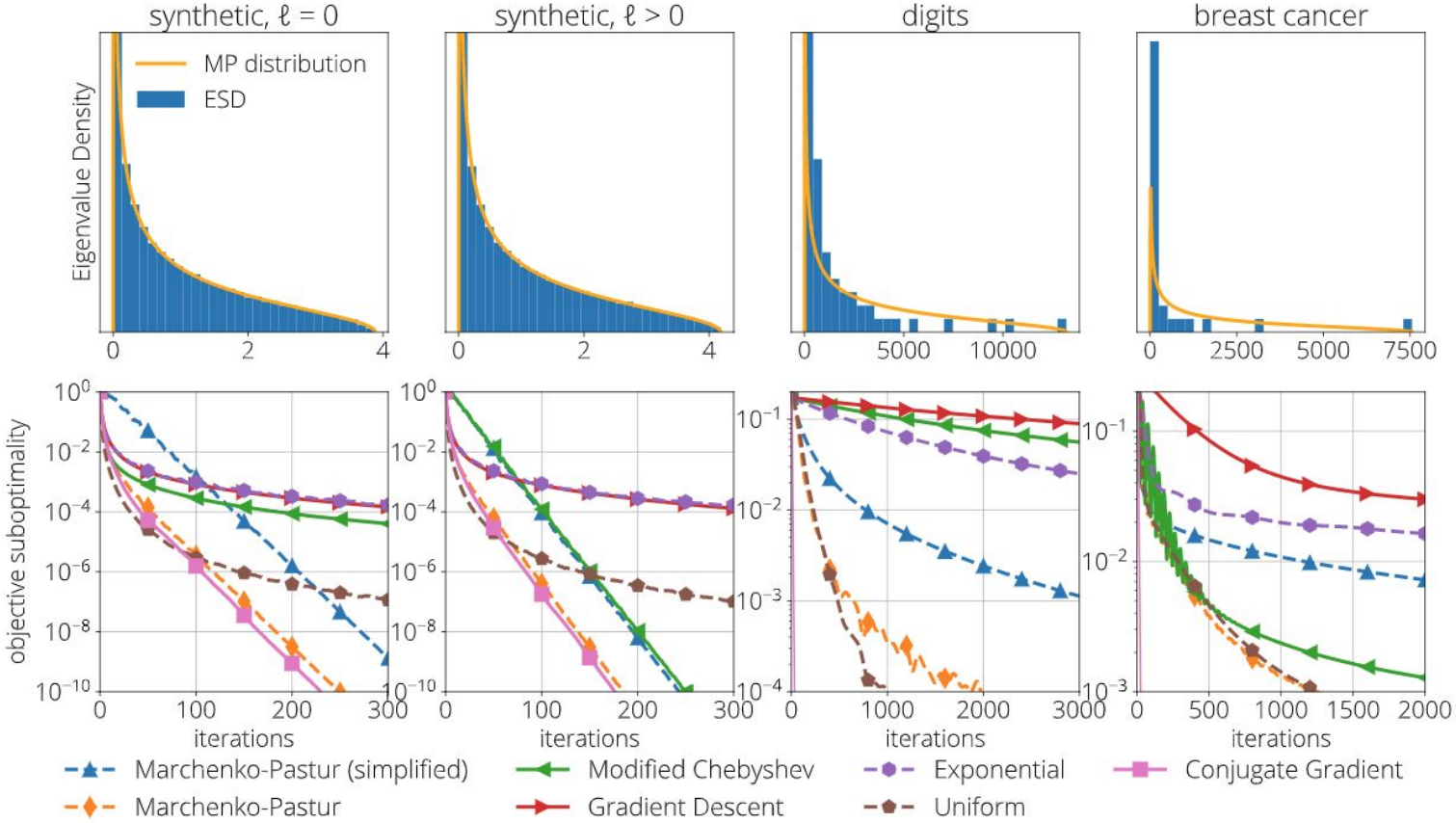
# Benchmarks: Least Squares



# Conclusions

Average-case analysis based on random quadratics.

Optimal methods under different eigenvalue distribution.

✓ Acceleration without knowledge of strong convexity.

In **paper**

\+ More methods, convergence rates, empirical extension to non-quadratic objectives.

**Follow-up work** on asymptotic analysis

(Scieur and P., *"Universal Average-Case Optimality of Polyak Momentum"*)

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