Random Walks on Hypergraphs with Edge-Dependent Vertex Weights

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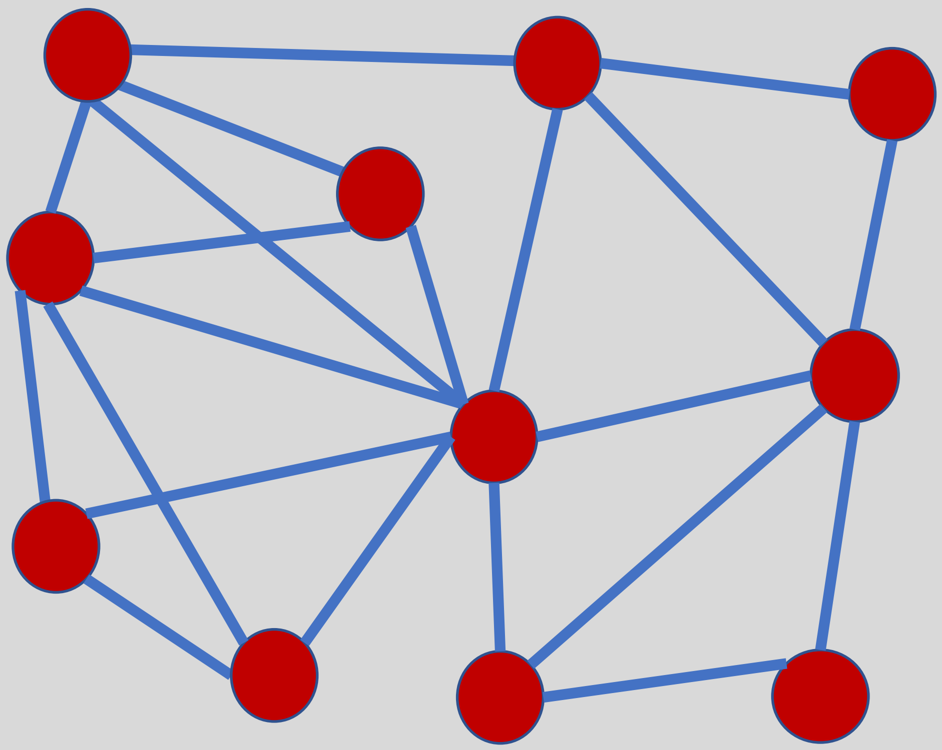
ICML 2019

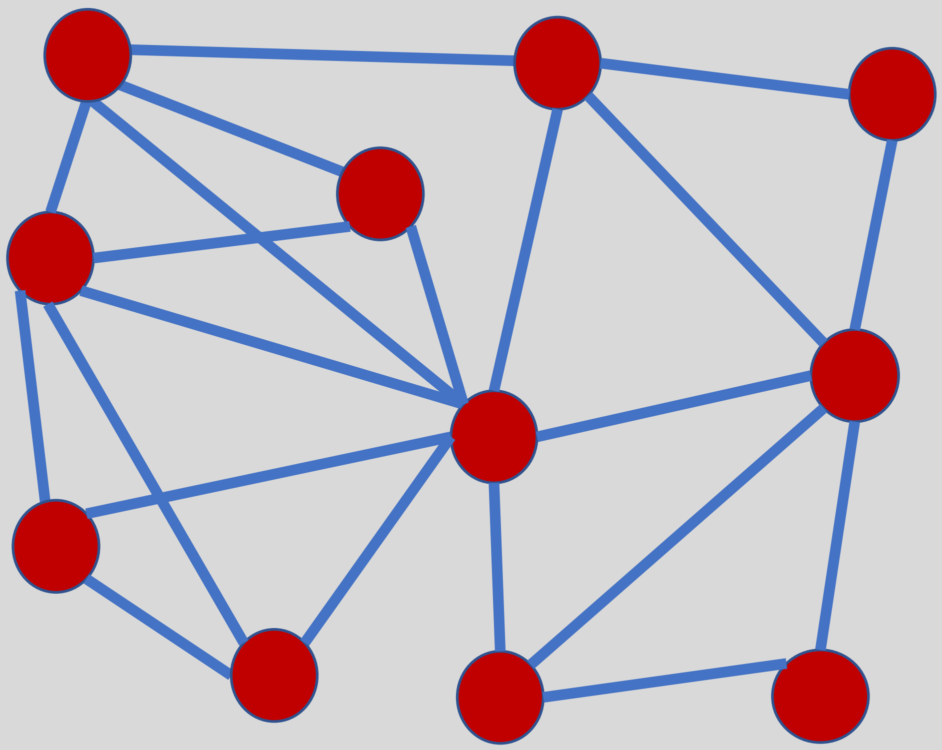
Graphs in Machine Learning



Graphs model pairwise relationships between objects

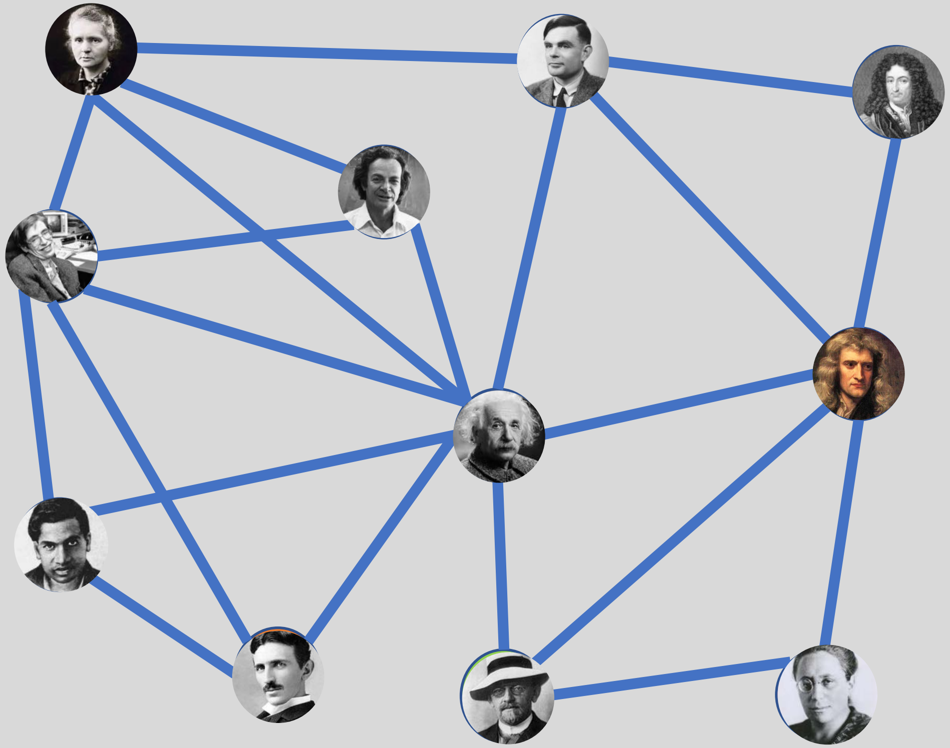
Examples:
- Social networks
- Internet
- Biological systems

Graphs in Machine Learning



However, graphs may <u>lose information</u> about the relationships between objects.

Graphs in Machine Learning

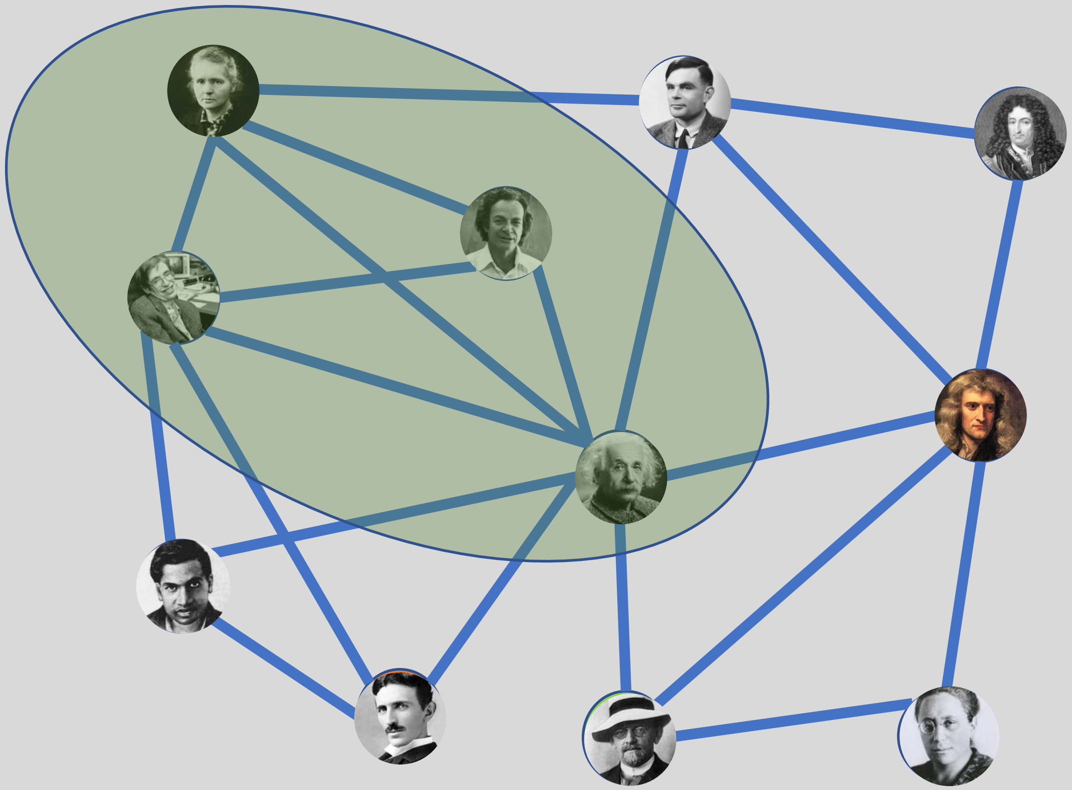


Above: a fictitious network of authors.

However, graphs may <u>lose information</u> about the relationships between objects.

Example: Given a co-authorship network, which authors wrote which papers?

Graphs in Machine Learning



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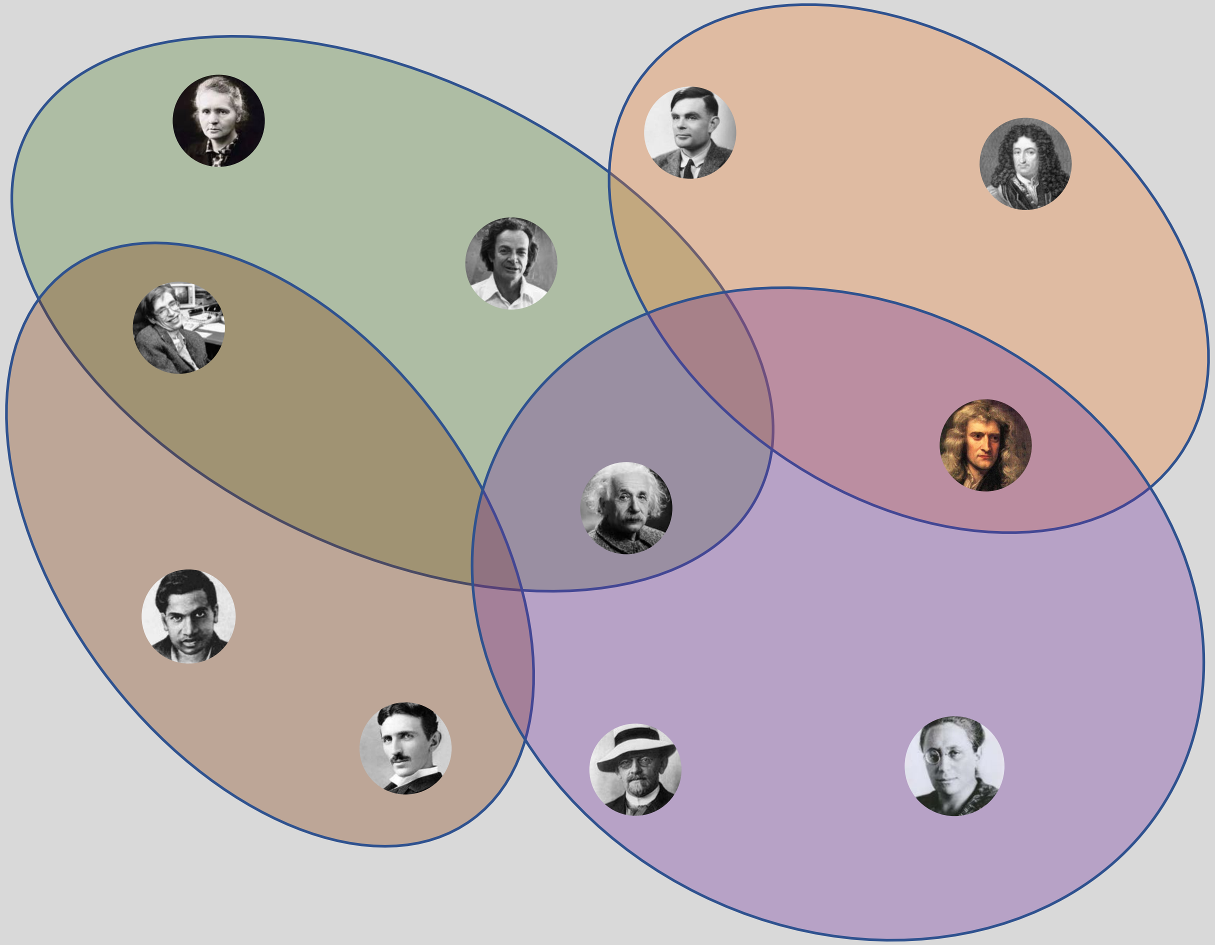
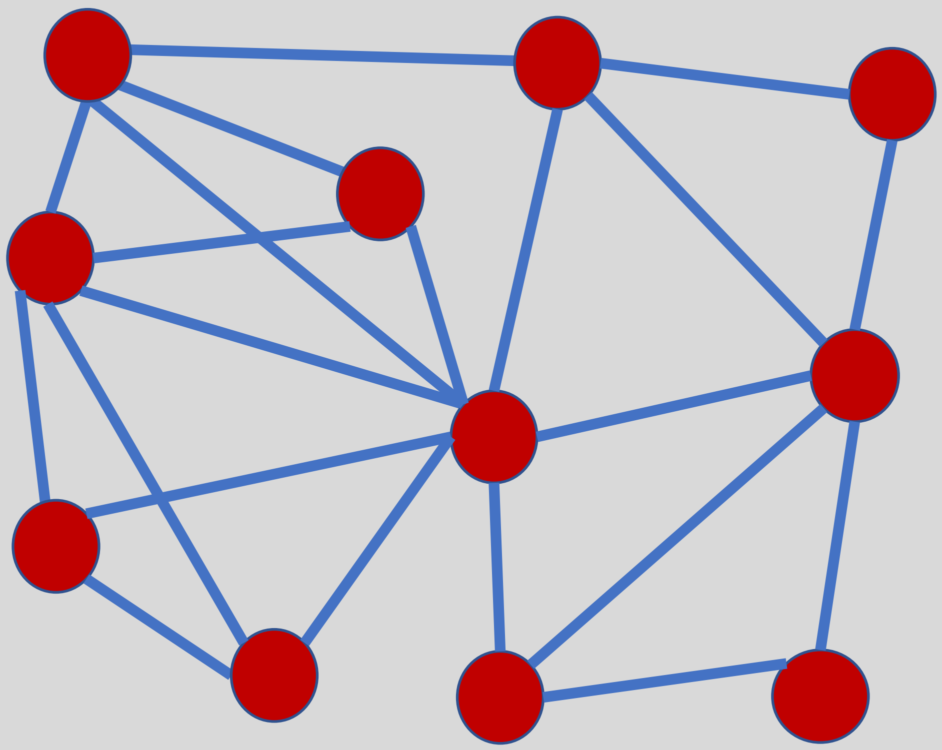
Hypergraphs in Machine Learning



A <u>hypergraph</u> $H = (V,E)$ models higher order relationships.

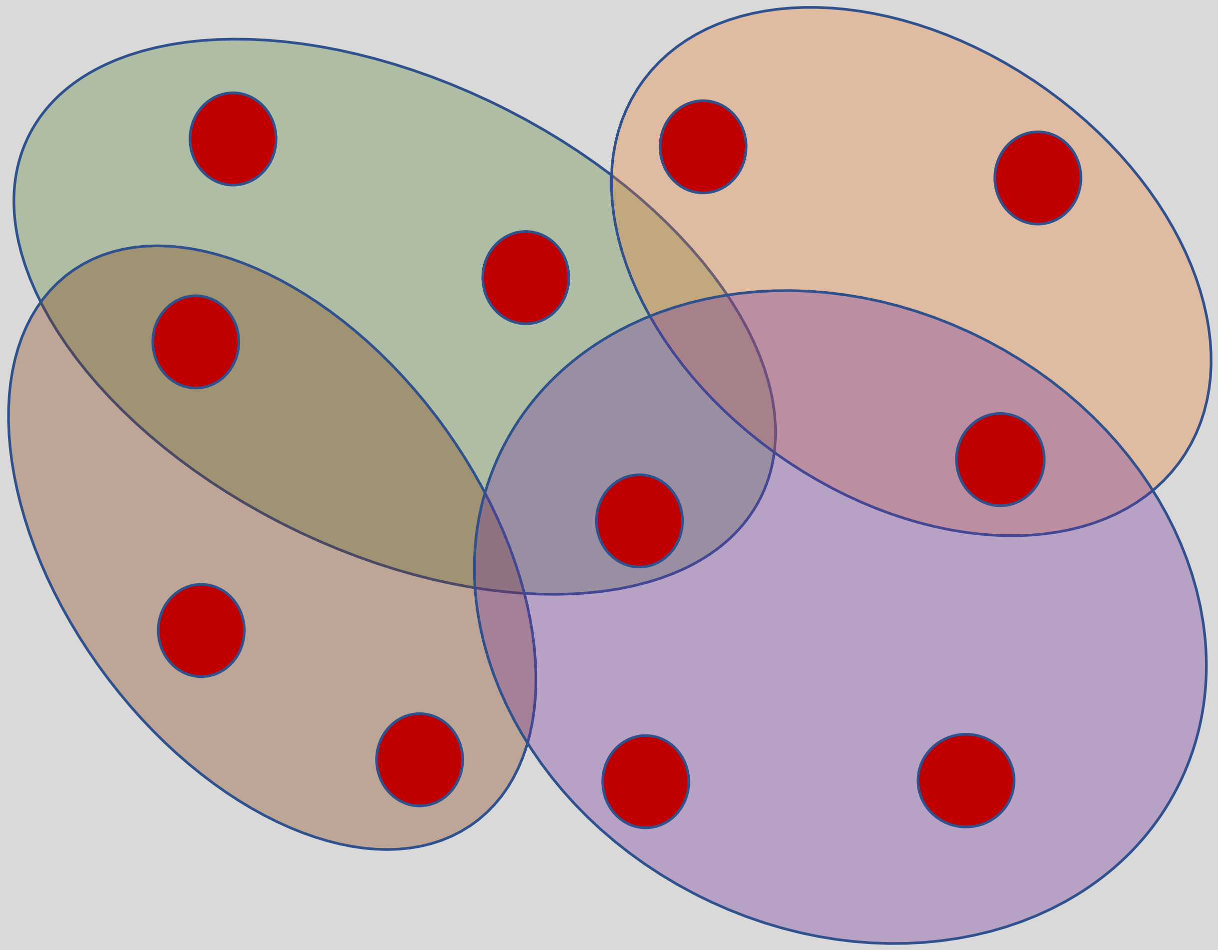
$E \subseteq 2^V$ is a set of hyperedges. Each hyperedge $e \in E$ can contain > 2 vertices.

Graphs



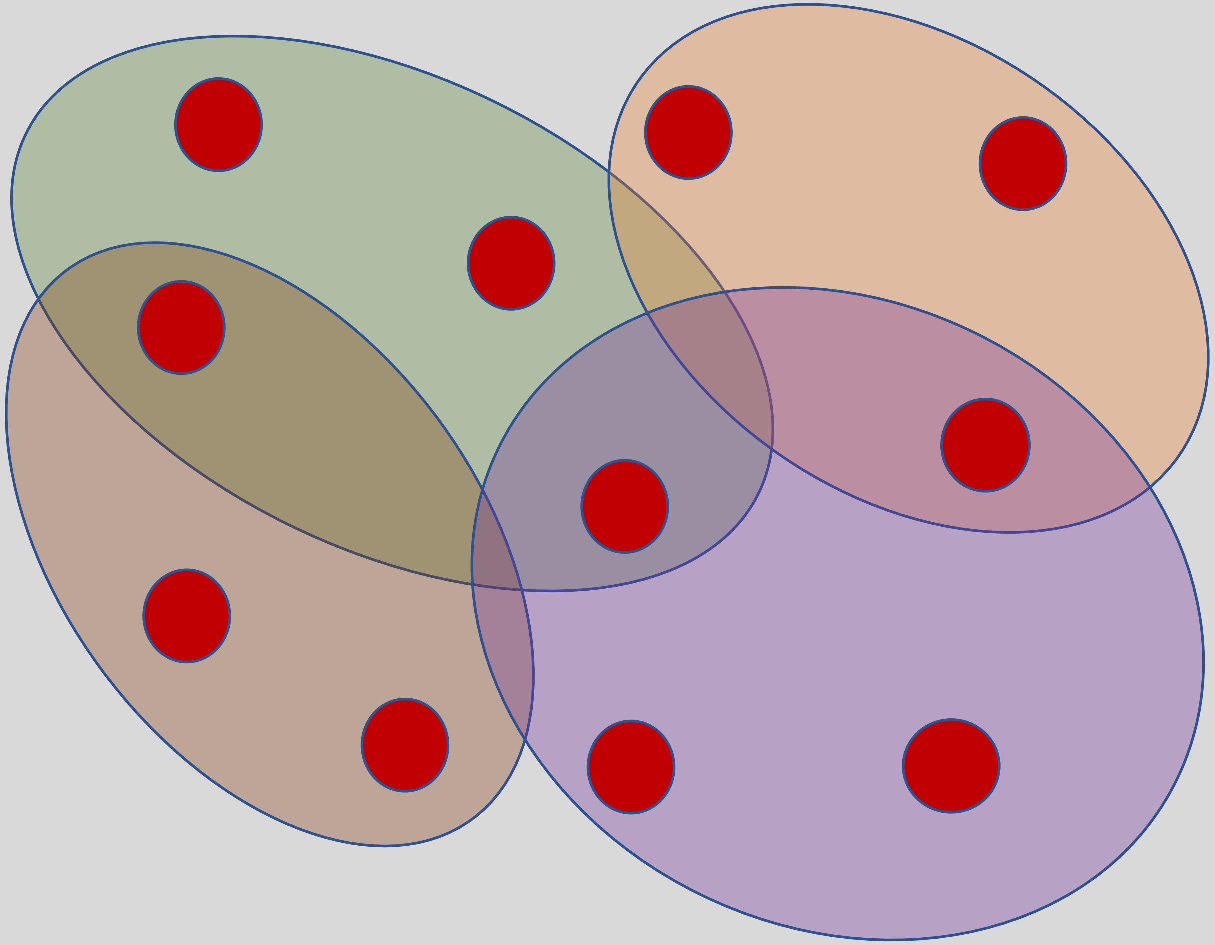
Model <u>pairwise</u> relationships

Hypergraphs



Model <u>higher-order</u> relationships

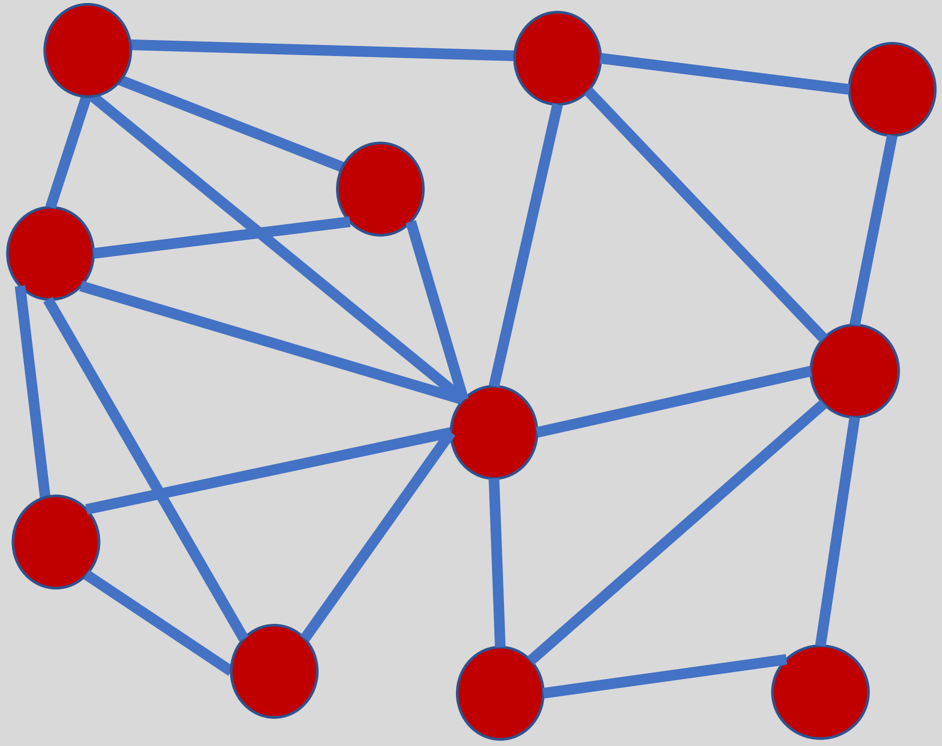
Hypergraphs in Machine Learning



Zhou, Huang, and Schölkopf [NeurIPS 2006]:

- Adapt spectral clustering methods to hypergraphs by defining a **hypergraph Laplacian matrix**
- demonstrate improvements over graphs in classification tasks

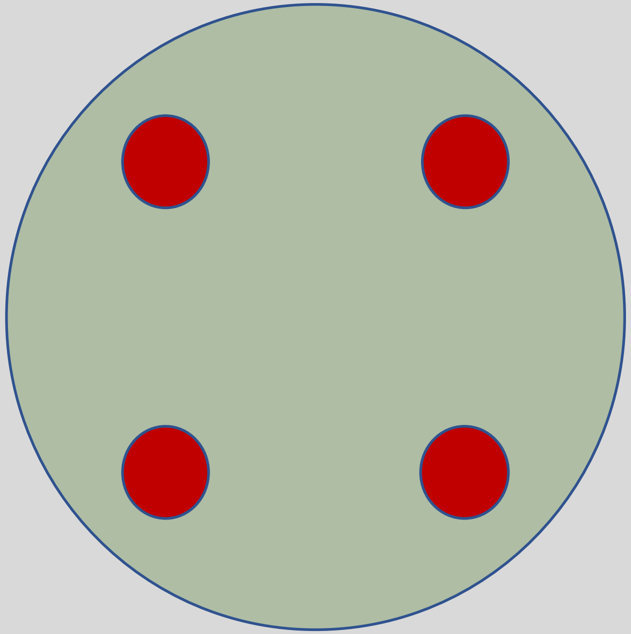
Do Hypergraphs Model Higher-Order Information?



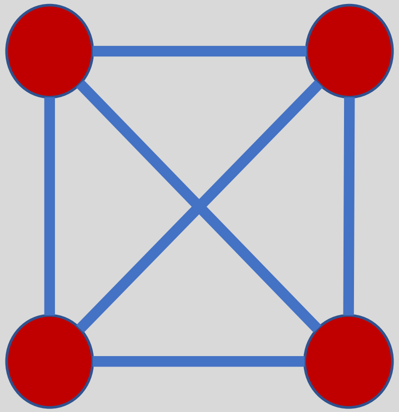
However, Agarwal et al. [ICML 2006] show that Zhou et al. are really doing inference on <u>graphs</u>

Do Hypergraphs Model Higher-Order Information?

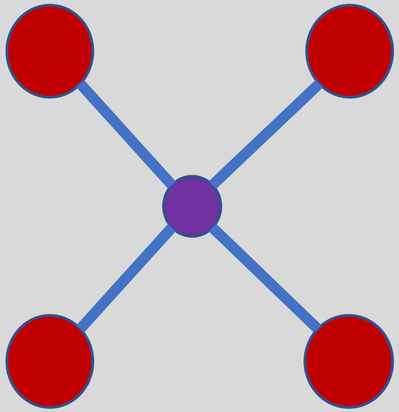
Specifically, Agarwal et al. shows that Zhou et al.'s hypergraph Laplacian matrix (and others in the literature) are equal to Laplacians of: either **clique** graph, or **star** graph



Hyperedge



Clique graph



Star graph

Do Hypergraphs Model Higher-Order Information?

Question: When do hypergraph learning algorithms not reduce to graph algorithms?

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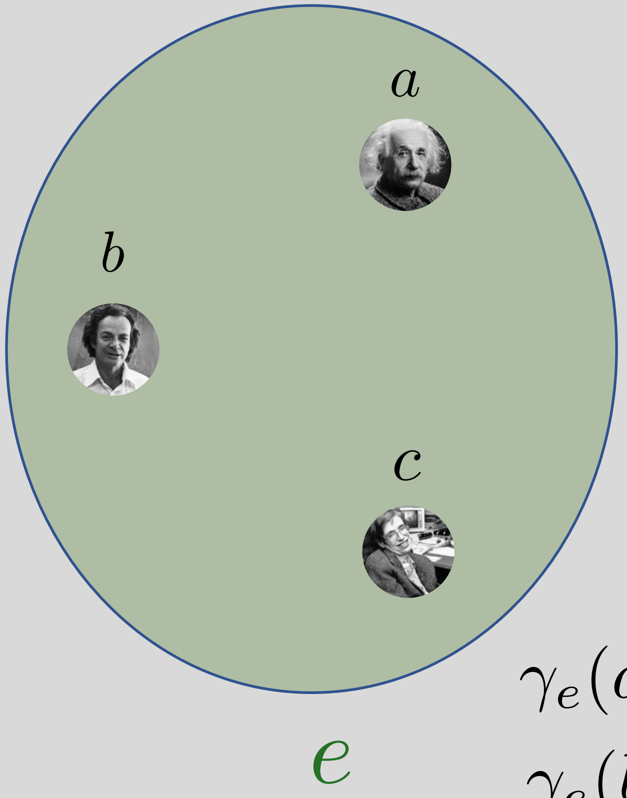
Our work: When the hypergraph has **edge-dependent vertex weights.**

What are Edge-Dependent Vertex Weights?

A vertex v has weight $\gamma_e(v)$ for each incident hyperedge e.

$\gamma_e(v)$ describes **the contribution of vertex v to hyperedge e.**

Example: in co-authorship network, edge-dependent vertex weights can measure the contribution of each author to a paper

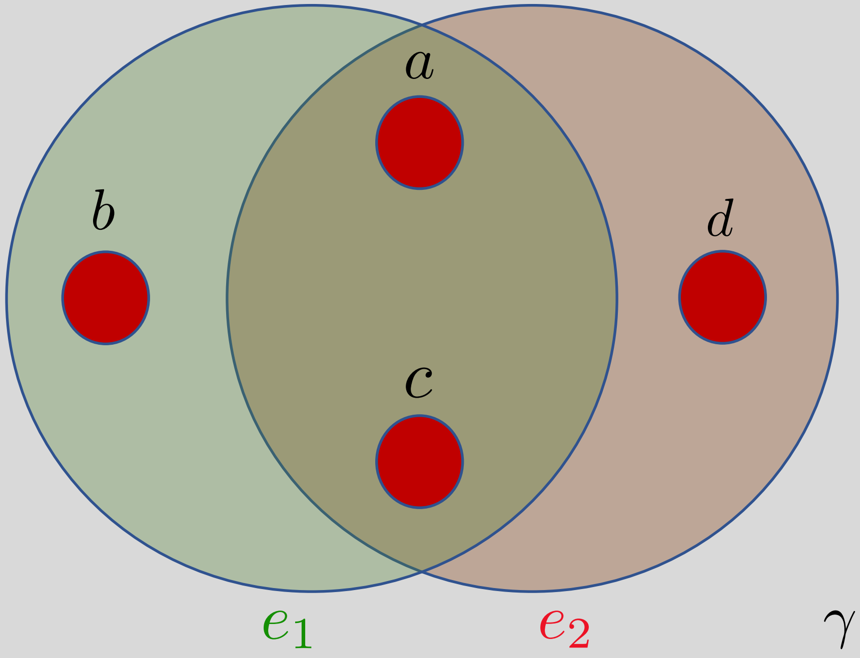


$$\gamma_e(a) = 5$$
$$\gamma_e(b) = 3$$
$$\gamma_e(c) = 1$$

Edge-Dependent vs Edge-Independent

In contrast, edge-*independent* vertex weights: $\gamma_e(v) = \gamma_f(v)$ for all hyperedges e, f incident to v

Most hypergraph literature assumes edge-independent vertex weights. (Typically the vertex weights are 1.)



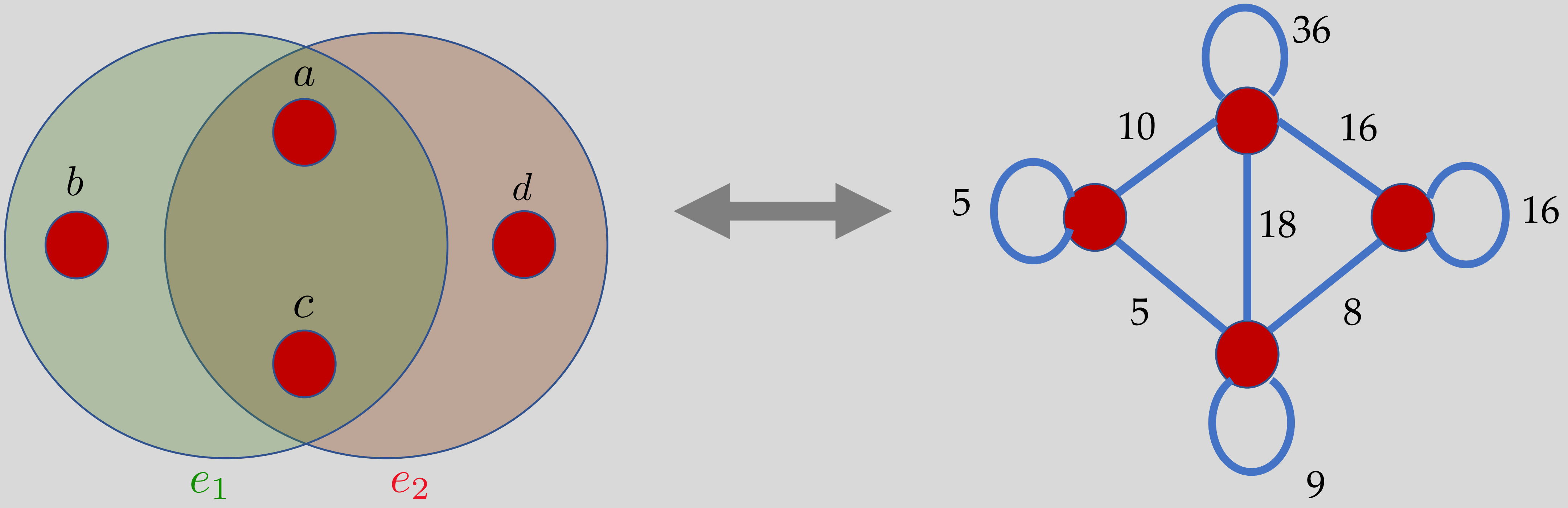
$$\gamma(a) = 2$$
$$\gamma(b) = 1$$
$$\gamma(c) = 1$$
$$\gamma(d) = 2$$

Part 1: Edge-Independent Vertex Weights

We show: When vertex weights are edge-independent, then random walks on hypergraph = random walks on clique graph

(Formally, the random walks have equal probability transition matrices)

$\gamma(a) = 2$
$\gamma(b) = 1$
$\gamma(c) = 1$
$\gamma(d) = 2$



Part 1: Edge-Independent Vertex Weights

Thus, existing hypergraph Laplacian matrices (e.g. Zhou et al.) are equal to Laplacian matrix of a clique graph

This is because these Laplacians are derived from random walks on hypergraphs with edge-independent vertex weights

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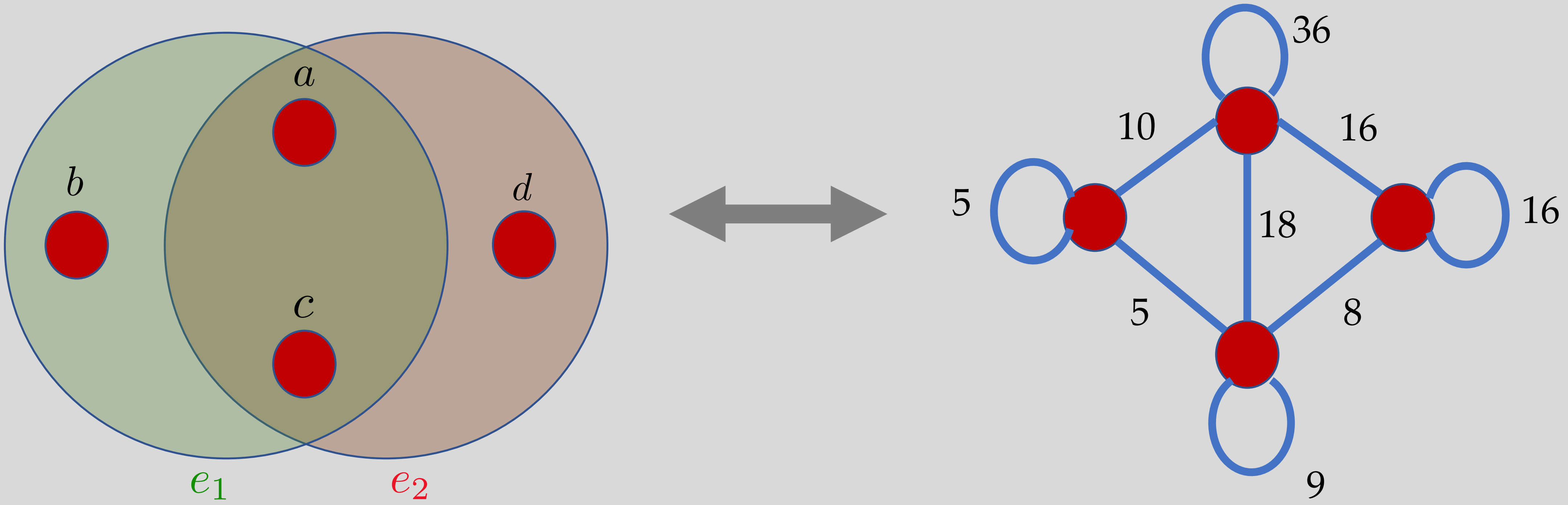
$\gamma(d) = 2$



Part 1: Edge-<u>Independent</u> Vertex Weights

Thus, existing hypergraph Laplacian matrices (e.g. Zhou et al.) are equal to Laplacian matrix of a clique graph

> This is because these Laplacians are derived from random walks on hypergraphs with edge-<u>independent</u> vertex weights

Generalizing Agarwal et al, we give the underlying reason that hypergraphs with edge-*independent* vertex weights do not utilize <u>higher-order relations</u> between objects

Part 2: Edge-Dependent Vertex Weights

Conversely, **we show** that random walks on hypergraphs with edge-dependent vertex weights ≠ random walks on clique graph.

Formally, there exists such a hypergraph whose random walk is not the same as a random walk on clique graph for *any* choice of edge weights

$\gamma_{e_1}(a) = 2$

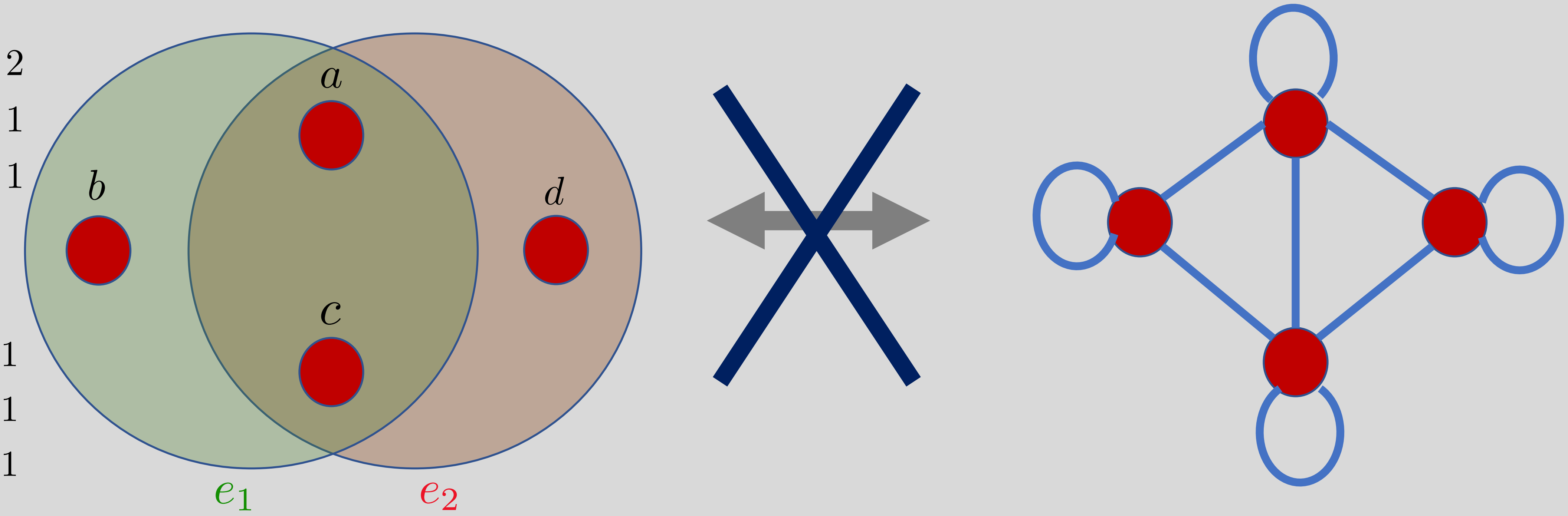
$\gamma_{e_1}(b) = 1$

$\gamma_{e_1}(c) = 1$

$\gamma_{e_2}(a) = 1$

$\gamma_{e_2}(c) = 1$

$\gamma_{e_2}(d) = 1$



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Thus, hypergraphs with edge-*dependent* vertex weights utilize higher-order relations between objects

Part 3: Theory for Edge-Dependent Vertex Weights

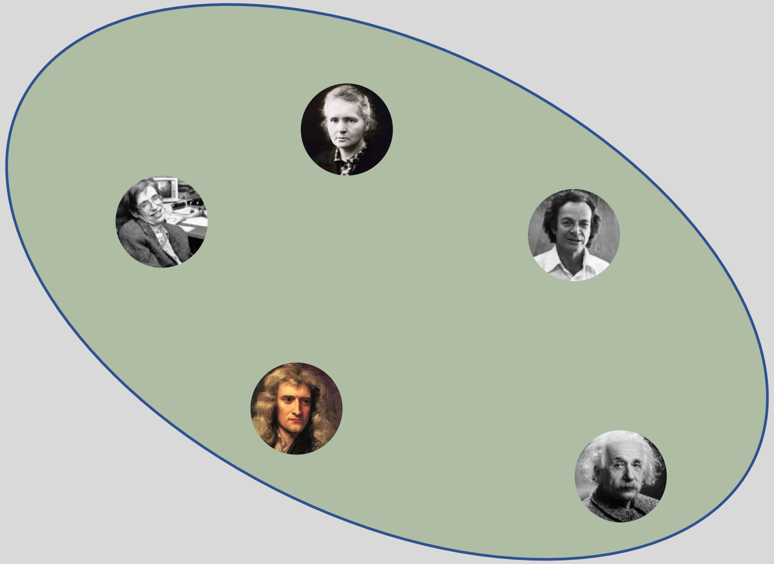
Motivated by this result, we develop a **spectral theory** for hypergraphs with edge-dependent vertex weights

	Graphs	Hypergraphs with **edge-dependent** vertex weights
Stationary distribution	$\pi_v = \rho \sum_{e \in E(v)} w(e)$	$\pi_v = \sum_{e \in E(v)} \rho_e \omega(e) \gamma_e(v)$
Mixing time of random walk	$t^G_{mix}(\epsilon) = \frac{2}{\Phi^2} \log\left(\frac{1}{2\epsilon\sqrt{d_{min}}}\right)$	$t^H_{mix}(\epsilon) = \frac{8\beta_1}{\Phi^2} \log\left(\frac{1}{2\epsilon\sqrt{d_{min}\beta_2}}\right)$
Laplacian matrix + Cheeger inequality	$L = D - A$	$L = \Pi - \frac{\Pi P + P^T \Pi}{2}$

Part 4: Experiments

We demonstrate two applications of edge-dependent vertex weights:

1. Ranking authors in citation network
2. Ranking players in a **multiplayer** video game



$$\gamma_e(v) = \begin{cases} 2 & \text{if vertex } v \text{ is the first or last author of paper,} \\ 1 & \text{if vertex } v \text{ is a middle author of paper.} \end{cases}$$

Thank you for listening!

Check out our poster: #216 at the Pacific Ballroom, tonight at 6:30 – 9pm

Our full paper is also in ICML 2019 proceedings and on arXiv.

Random Walks on Hypergraphs with Edge-Dependent Vertex Weights

Uthsav Chitra, Benjamin J Raphael

(Submitted on 20 May 2019)

Hypergraphs are used in machine learning to model higher-order relationships in data. While spectral methods for graphs are well-established, spectral theory for hypergraphs remains an active area of research. In this paper, we use random walks to develop a spectral theory for hypergraphs with edge-dependent vertex weights: hypergraphs where every vertex v has a weight $\gamma_e(v)$ for each incident hyperedge e that describes the contribution of v to the hyperedge e. We derive a random walk-based hypergraph Laplacian, and bound the mixing time of random walks on such hypergraphs. Moreover, we give conditions under which random walks on such hypergraphs are equivalent to random walks on graphs. As a corollary, we show that current machine learning methods that rely on Laplacians derived from random walks on hypergraphs with edge-independent vertex weights do not utilize higher-order relationships in the data. Finally, we demonstrate the advantages of hypergraphs with edge-dependent vertex weights on ranking applications using real-world datasets.

arXiv link: www.arxiv.org/abs/1905.08287