Multi-Frequency Phase Synchronization

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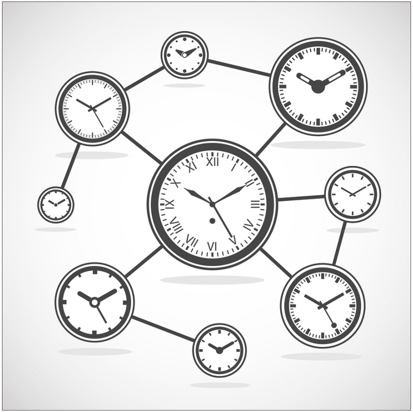
Phase Synchronization

- **Problem:** Recover rotation angles $\theta_1, \ldots, \theta_n \in [0, 2\pi]$ from noisy measurements of their pairwise offsets

$$\theta_{ij} = \theta_i - \theta_j + \text{noise}$$

for some or all pairs of (i, j)

- **Examples:** Class averaging in cryo-EM image analysis, shape registration and community detection



Phase Synchronization

- **Setup**: Phase vector $z = \left(e^{\iota\theta_1}, \ldots, e^{\iota\theta_n}\right)^\top \in \mathbb{C}_1^n$, noisy pairwise measurements in n-by-n Hermitian matrix

$$H_{ij} = \begin{cases} e^{\iota\left(\theta_i - \theta_j\right)} = z_i \bar{z}_j & \text{with prob. } r \in [0,1] \\ \mathrm{Uniform}\left(\mathbb{C}_1\right) & \text{with prob. } 1 - r \end{cases}$$

and $H_{ij} = \overline{H_{ji}}$. This is known as a random corruption model.

- **Goal**: recover the true phase vector z (up to a global multiplicative factor)
- **Existing method**: Rank-1 recovery (e.g. convex relaxations)

$$\hat{x} := \arg\min_{x \in \mathbb{C}_1^n} \|xx^* - H\|_{\mathrm{F}}^2 \quad \Leftrightarrow \quad \hat{x} := \arg\max_{x \in \mathbb{C}_1^n} x^* H x$$

Multi-Frequency Phase Synchronization

- **Multi-Frequency Formulation:**

$$\boxed{\max_{x \in \mathbb{C}_1^n} \sum_{k=1}^{k_{\max}} (x^k)^* H^{(k)} x^k}$$

where $x^k := \left(x_1^k, \ldots, x_n^k\right)^\top \in \mathbb{C}_1^n$, and $H^{(k)}$ is the n-by-n Hermitian matrix with $H_{ij}^{(k)} := H_{ij}^k$

- **Intuition:** Matching higher trigonometric moments
- **Two-stage Algorithm:** (i) Good initialization (ii) Local methods e.g. gradient descent or (generalized) power iteration

Initialization: Inspired by Harmonic Retrieval

- Fix $k_{\max} \geq 1$, build $H^{(2)}, \ldots, H^{(k_{\max})}$ out of $H = H^{(1)}$
- For each $k = 1, \ldots, k_{\max}$, solve the subproblem

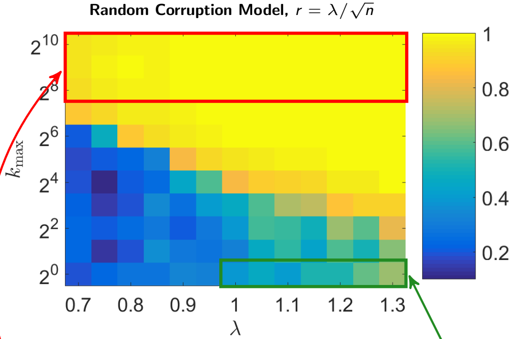
$$u^{(k)} := \arg\max_{v \in \mathbb{C}_1^n} \; v^* H^{(k)} v$$

  using any convex relaxation, and set $W^{(k)} := u^{(k)} \left(u^{(k)}\right)^*$
- For all $1 \leq i, j \leq n$, find the "peak location" of the spectrogram

$$\hat{\theta}_{ij} := \arg\max_{\phi \in [0, 2\pi]} \left| \frac{1}{2} \sum_{k=-k_{\max}}^{k_{\max}} W_{ij}^{(k)} e^{-\iota k \phi} \right|$$

- Entrywise normalize the top eigenvector $\tilde{x}$ of Hermitian matrix $\widehat{H}$, defined by $\widehat{H}_{ij} = e^{\iota \hat{\theta}_{ij}}$, to get $\hat{x} \in \mathbb{C}_1^n$

How well does it work? Evaluate correlation $|\mathrm{Corr}(\hat{x}, z)|$



Our Method: $|\mathrm{Corr}(\hat{x}, z)| \longrightarrow 1$ as $k_{\max} \gg 1$, **even for** $\lambda < 1$!

Previous Art: Only ensures $|\mathrm{Corr}(\hat{x}, z)| > \frac{1}{\sqrt{n}}$ **for** $\lambda > 1$

Grounded Upon Solid Theory

Theorem (Gao & Zhao 2019). With all (mild) assumptions satisfied, with high probability the multi-frequency phase synchronization algorithm produces an estimate $\hat{x}$ satisfying

$$\mathrm{Corr}\left(\hat{x}, z\right) \geq 1 - \frac{C'}{k_{\mathrm{max}}^2}$$

provided that

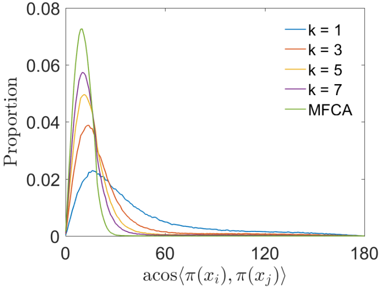
$$k_{\mathrm{max}} > \max\left\{5, \frac{1}{\sqrt{2}\,\pi\left(1 - 4C_2\sigma\sqrt{\log n/n}\right) - 2}\right\}.$$

In particular, $\mathrm{Corr}\left(\hat{x}, z\right) \to 1$ as $k_{\mathrm{max}} \to \infty$.

• Tingran Gao and Zhizhen Zhao, "Multi-Frequency Phase Synchronization." Proceedings of the 36th International Conference on Machine Learning, PMLR 97:2132–2141, 2019.

Thank You!

Poster **Today**: 06:30–09:00PM **@** Pacific Ballroom #143



- Tingran Gao and Zhizhen Zhao, "Multi-Frequency Phase Synchronization." Proceedings of the 36th International Conference on Machine Learning, PMLR 97:2132–2141, 2019.
- Tingran Gao, Yifeng Fan, and Zhizhen Zhao. "Representation Theoretic Patterns in Multi-Frequency Class Averaging for Three-Dimensional Cryo-Electron Microscopy," *arxiv:1906.01082*.