Composable Core-sets for Determinant Maximization: A Simple Near-Optimal Algorithm

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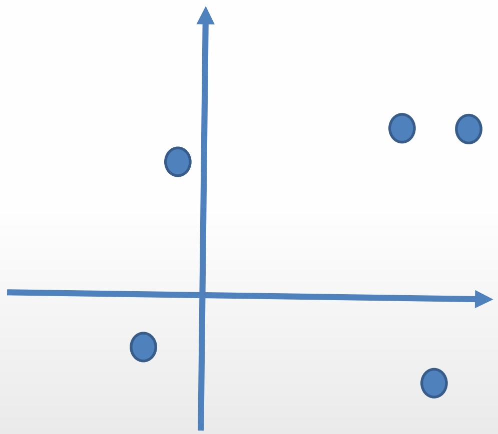
Volume (Determinant) Maximization Problem

Input: a set of n vectors $\mathrm{V} \in \mathbb{R}^d$ and a parameter $k \leq d$,

$\boldsymbol{k = 2}$



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Input: a set of n vectors $\mathrm{V} \in \mathbb{R}^d$ and a parameter $k \leq d$,

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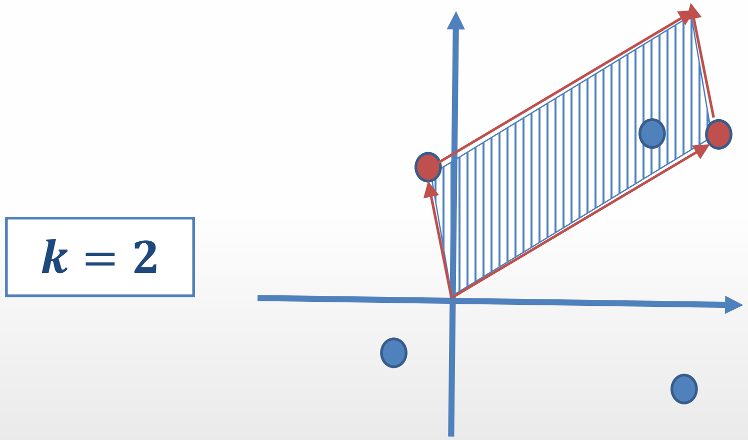
- Parallelepiped spanned by the points in S



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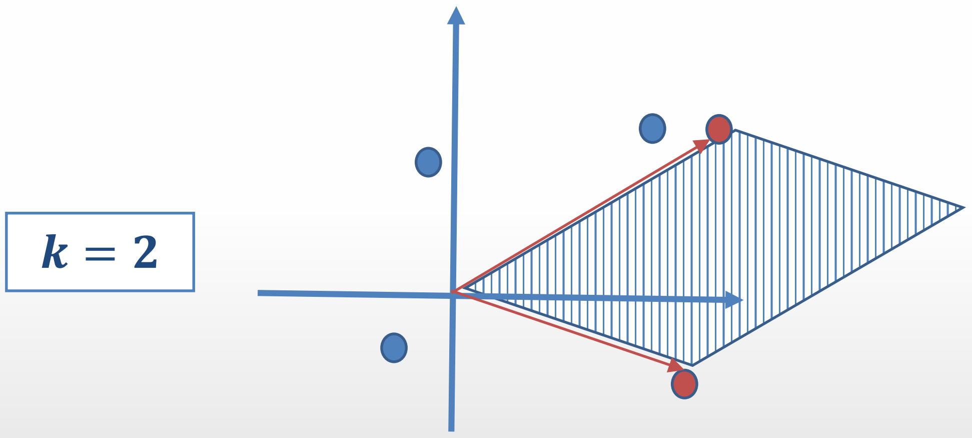

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$$\begin{bmatrix} v_1 \; v_2 \ldots v_n \end{bmatrix}$$

$$\boldsymbol{V}$$

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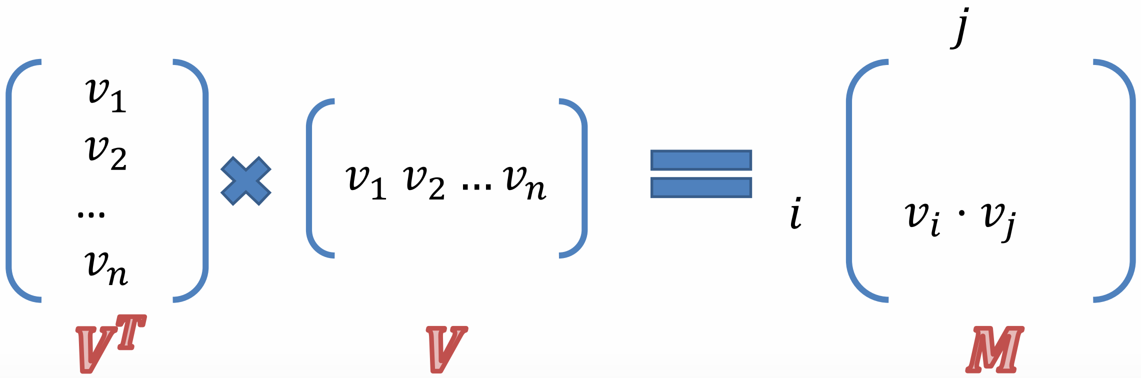

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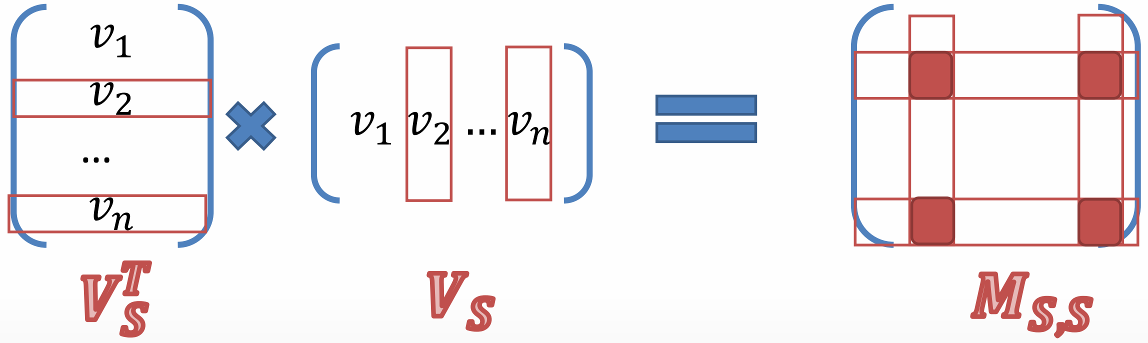

Equivalent Formulation:

Reuse V to denote the matrix where its columns are the vectors in V

- Let M be the gram matrix $V^T V$
- Choose S such that $\det(M_{S,S})$ is maximized

$M_{i,j} = v_i \cdot v_j$

$\det(M_{S,S}) = Vol(S)^2$

What is known?

- Hard to approximate within a factor of 2^{ck} [CMI'13]

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  - $U \leftarrow \emptyset$
  - For k iterations,
    - Add to U the farthest point from the subspace spanned by U

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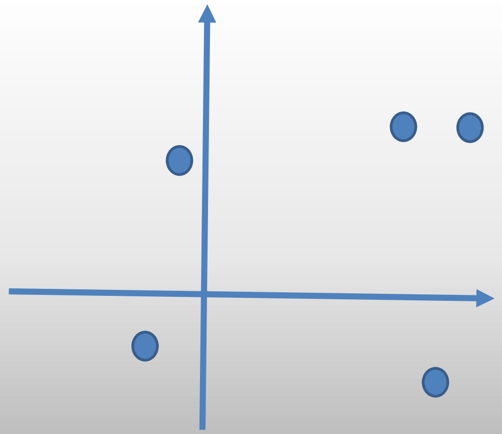

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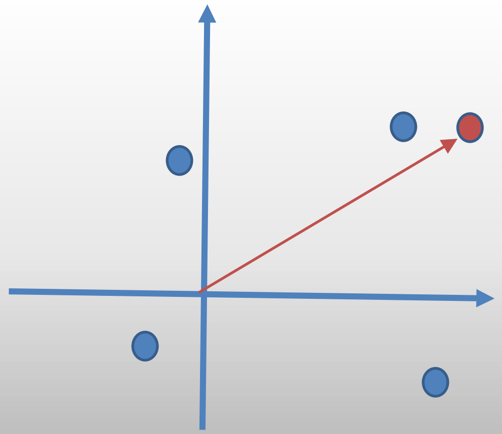

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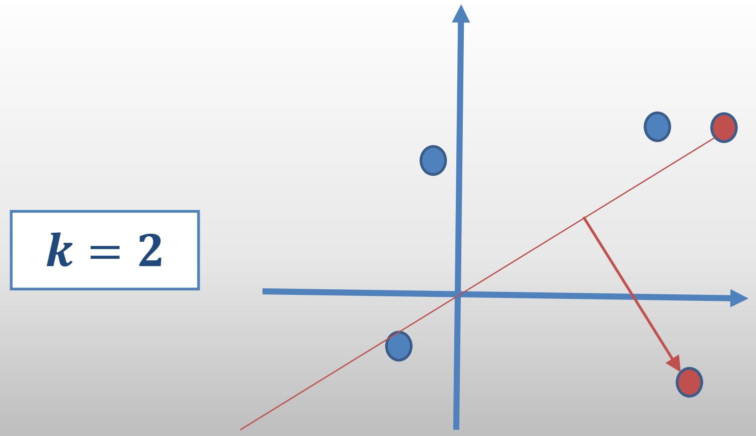

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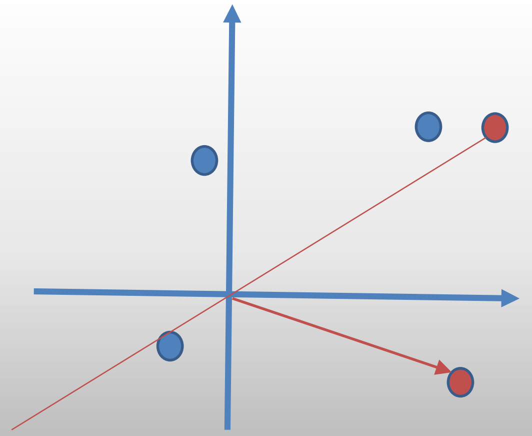

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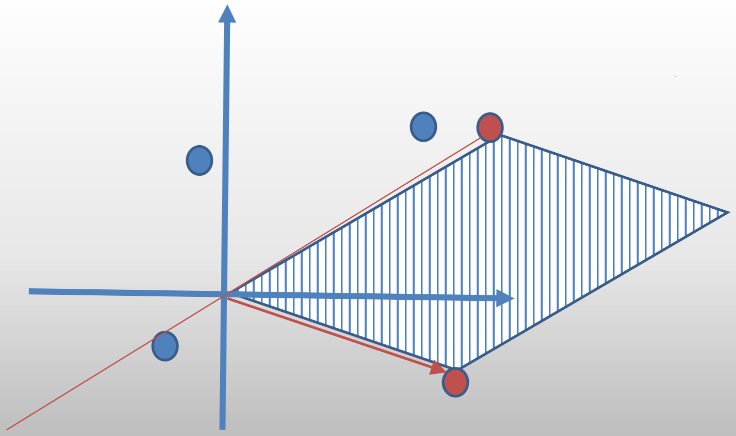

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- Greedy performs very well in practice

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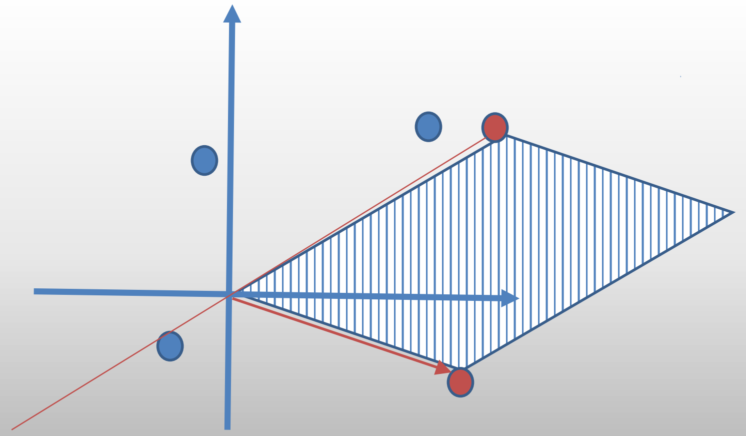

Determinantal Point Processes (DPP)

DPP: Very popular probabilistic model, where given a set of vectors V, **samples** any k-subset S with probability proportional to this determinant.

- Maximum a posteriori (MAP) decoding is determinant maximization
- Volume/determinant is a notion of **diversity**

- **NeurIPS'18 Tutorial,** *Negative Dependence, Stable Polynomials, and All That*, Jegelka, Sra
- **ICML'19 Workshop,** *Negative Dependence: Theory and Applications in Machine Learning,* Gartrell, Gillenwater, Kulesza, Mariet

Application: Diversity Maximization

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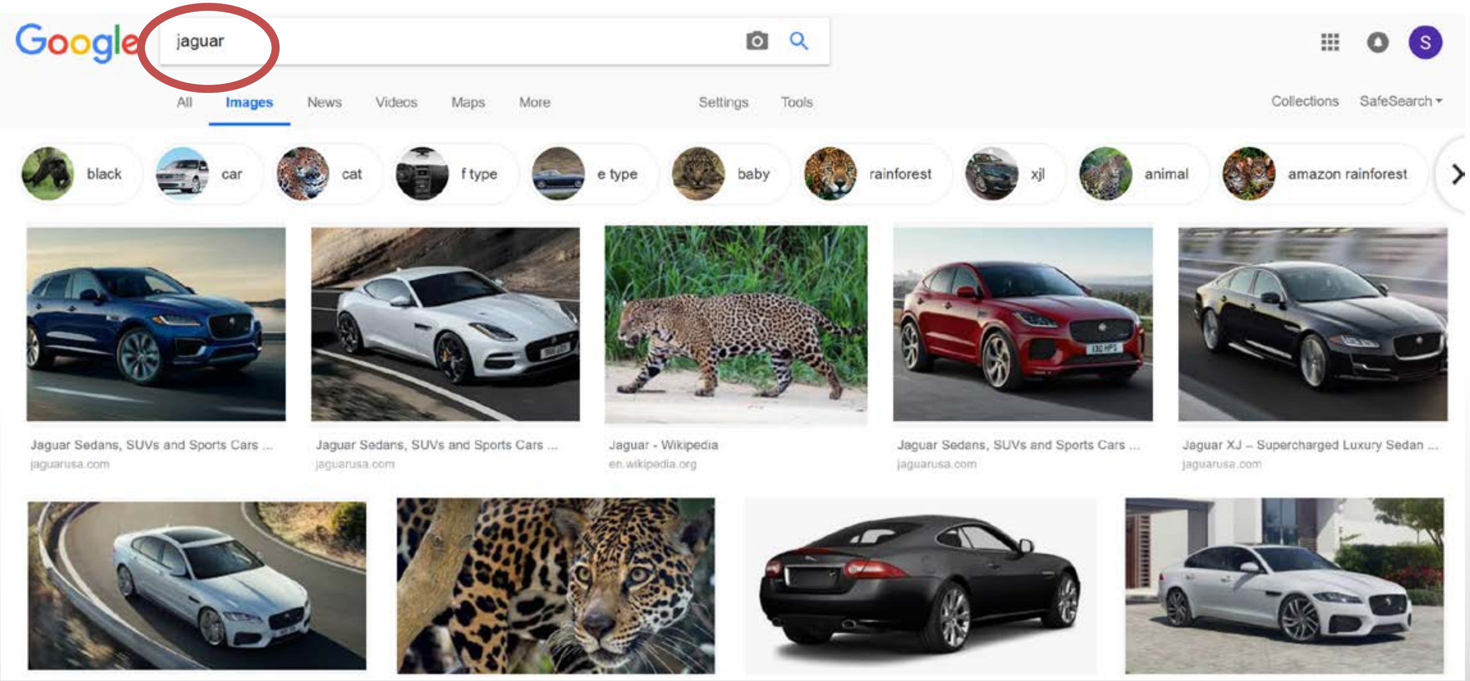

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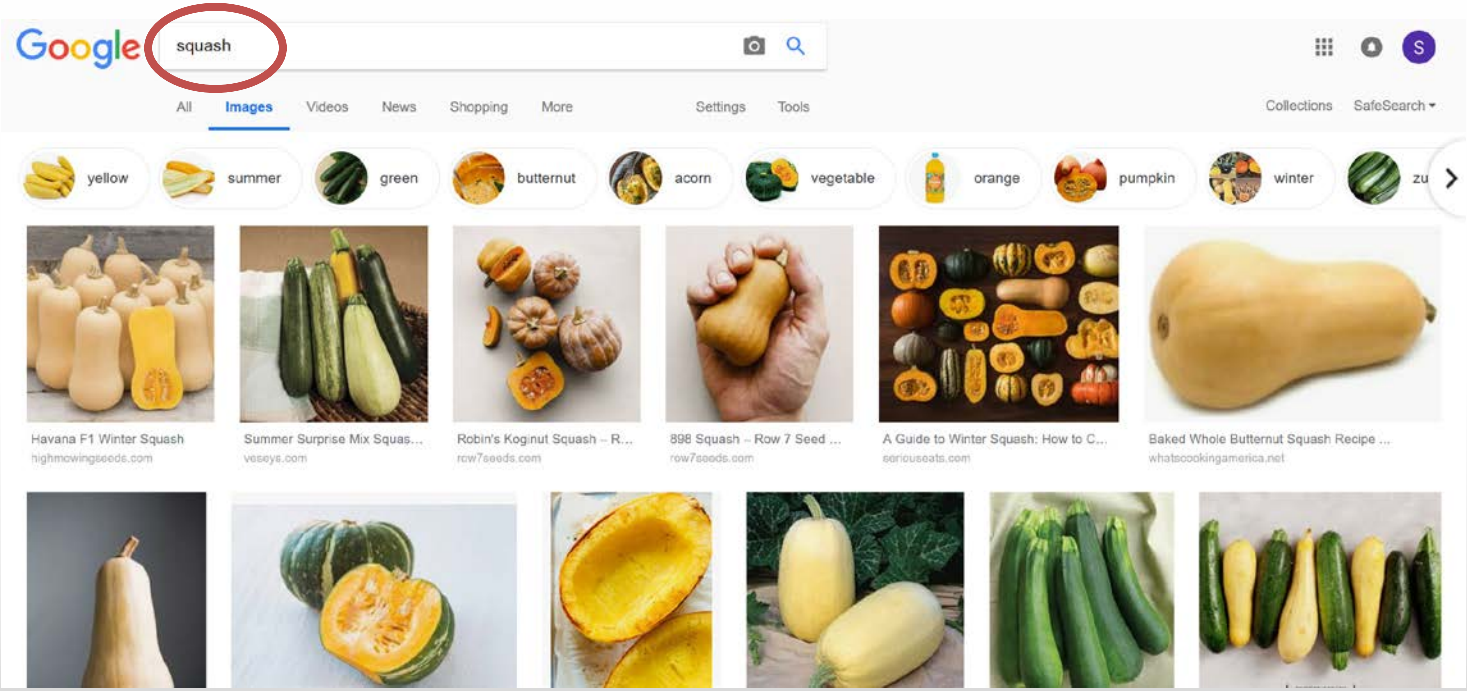
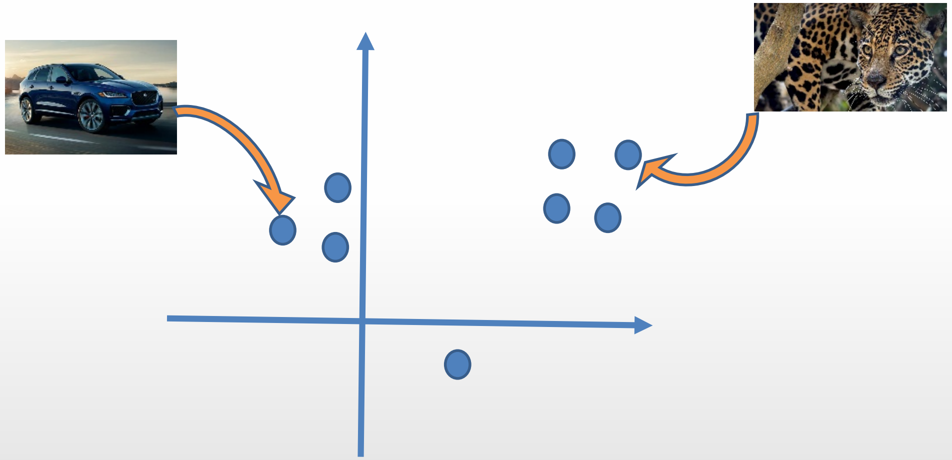


Application: Diversity Maximization

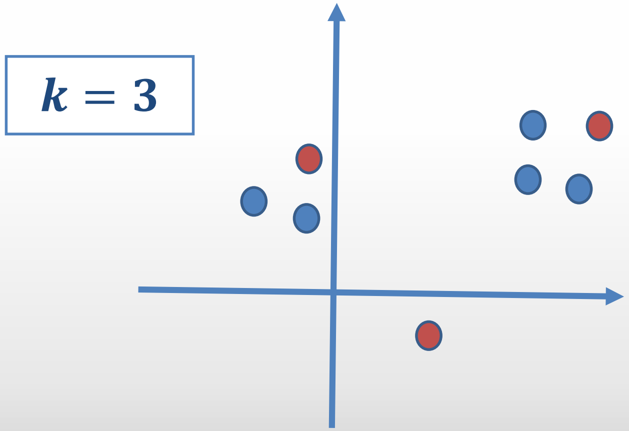
Objects (documents, images, etc) → Feature Vectors → Points in a high dimensional space



Application: Diversity Maximization

Input: a set of n vectors $V \subset \mathbb{R}^d$ and a parameter k,

Goal: **pick k points while maximizing "diversity".**



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- Most applications deal with **massive data**
- Lots of effort for solving the problem in massive data models of computation [MJK'17, WIB'14, PJG+'14, MKSK'13, MKBK'15, MZ'15, MZ'15, BENW'15]
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Composable Core-sets

Core-sets

Core-sets [AHV'05]**:** a subset $\boldsymbol{U}$ of the data $\boldsymbol{V}$ that represents it well

Solving the problem over $\boldsymbol{U}$ gives a good approximation of solving the problem over $\boldsymbol{V}$

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Core-sets [AHV’05]**:** a subset $\boldsymbol{U}$ of the data $\boldsymbol{V}$ that represents it well

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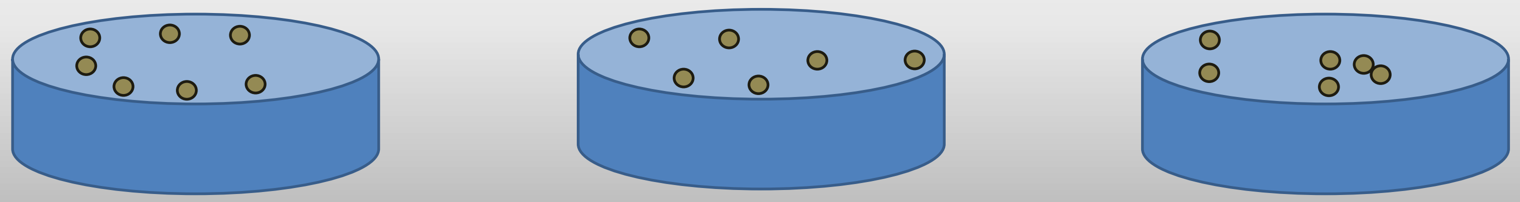

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- Let f be an optimization function
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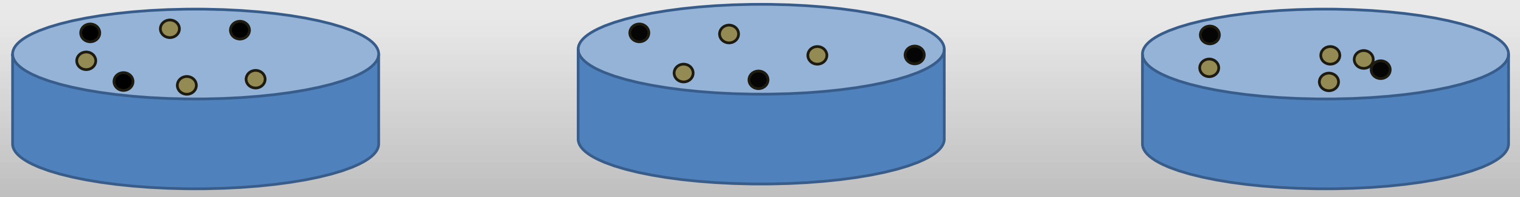

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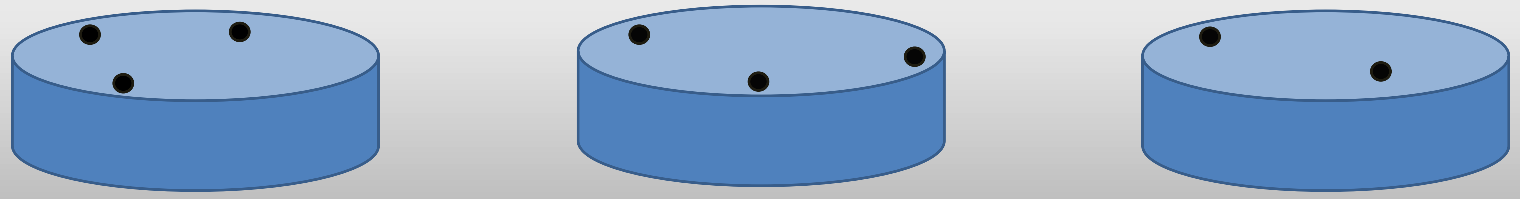

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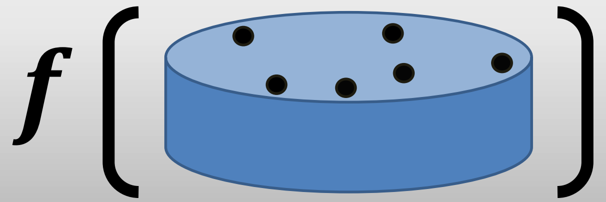

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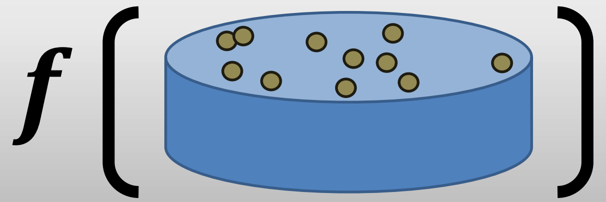

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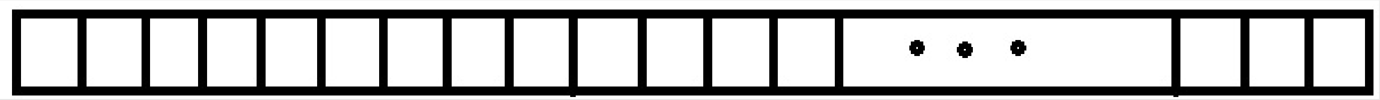
✓ Composable Core-sets have been studied for the **diversity Maximization** problems, for other notions of diversity: minimum pairwise distance, sum of pairwise distances, etc.

✓ Determinant maximization is a "higher order" notion of diversity

Applications: Streaming Computation

- **Streaming Computation:**
  - Processing sequence of n data elements "on the fly"
  - limited Storage



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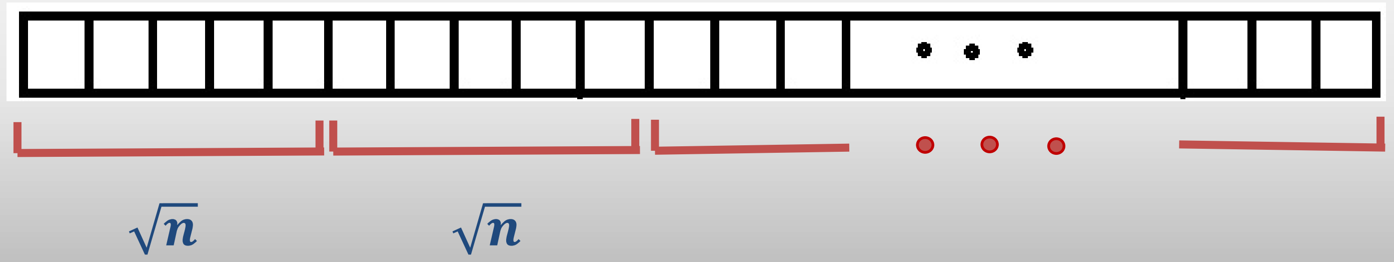

Applications: Streaming Computation

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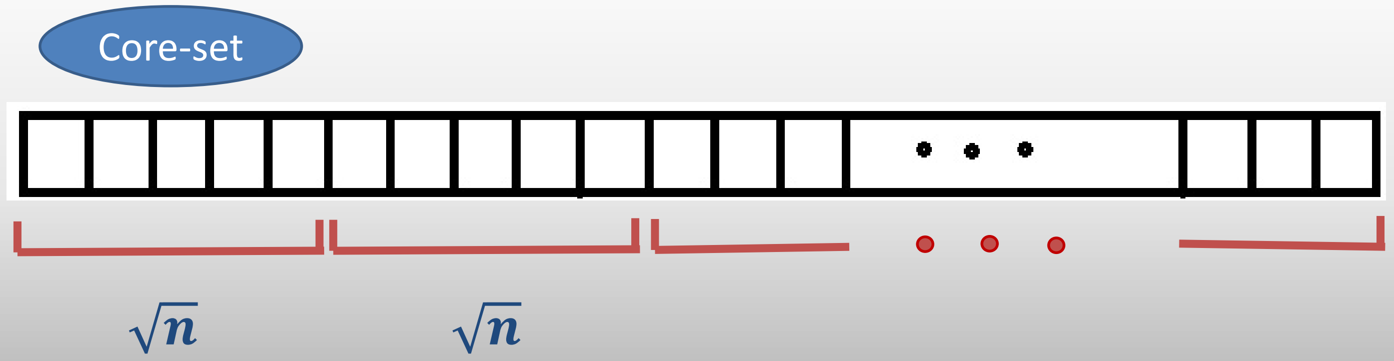
  - Compute Core-set for each chunk as it arrives



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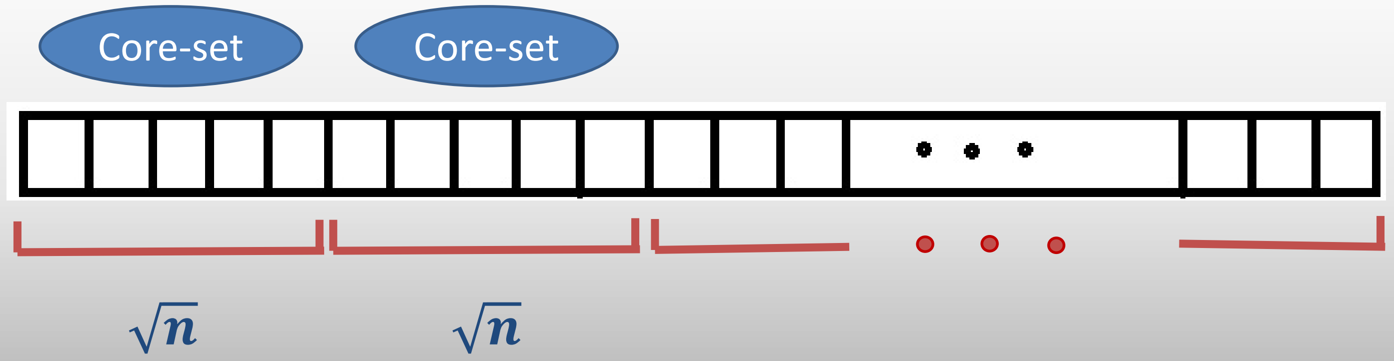

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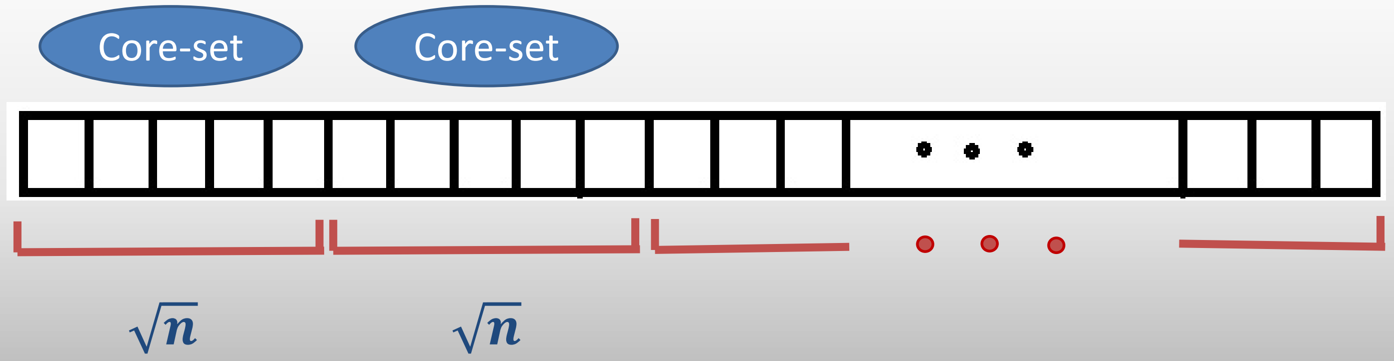
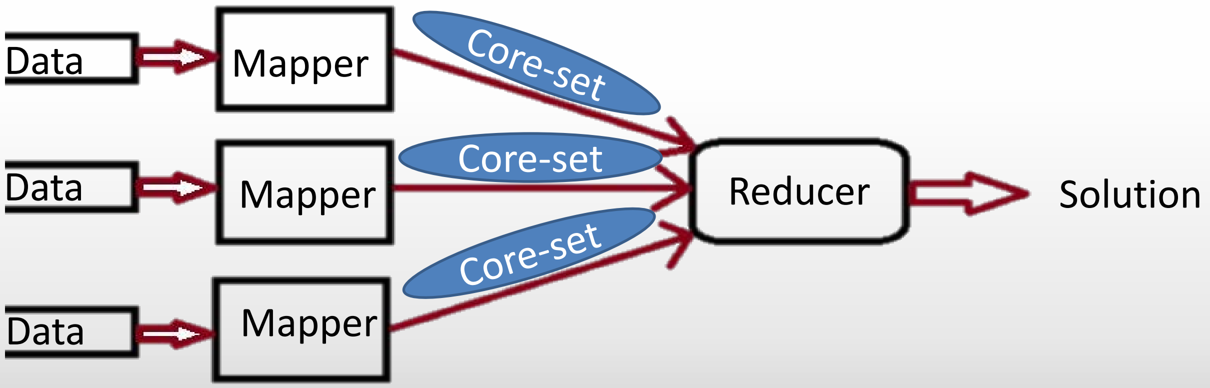
  - Space goes down from n to $\sqrt{n}$



Applications: Distributed Computation

- Streaming Computation
- **Distributed System:**
  - Each machine holds a block of data.
  - A composable core-set is computed and sent to the server



Applications: Improving Runtime

- Streaming Computation
- Distributed System
- **Similar framework for improving the runtime**

Can we get a composable core-set of small size for the determinant maximization problem?

Composable Core-sets for Volume Maximization

		[IMOR'18]
Approximation		$\tilde{O}(k)^{k/2}$
Core-set Size		$\tilde{O}(k)$
Simple?		×

LP-based Optimal Approximation Algorithm of [IMOR'18]:

There exists a polynomial time algorithm for computing an $\tilde{O}(k)^{k/2}$ -composable core-set of size $\tilde{O}(k)$ for the volume maximization problem.

Composable Core-sets for Volume Maximization

	Lower Bound	[IMOR'18]
Approximation	$\Omega(k)^{\frac{k}{2}-o(k)}$	$\tilde{O}(k)^{\frac{k}{2}}$
Core-set Size	$k^{O(1)}$	$\tilde{O}(k)$
Simple?		×

Lower bound [IMOR'18]:

Any composable core-set of size $k^{O(1)}$ for the volume maximization problem must have an approximation factor of $\Omega(k)^{\frac{k}{2}(1-o(1))}$.

Our Results

	Lower Bound	[IMOR'18]	Greedy
Approximation	$\Omega(k)^{\frac{k}{2}-o(k)}$	$\widetilde{O}(k)^{\frac{k}{2}}$	$O(C^{k^2})$
Core-set Size	$k^{O(1)}$	$\widetilde{O}(k)$	k
Simple?		×	✓

The widely used Greedy algorithm produces a composable core-set of size k with approximation factor $O(C^{k^2})$.

Our Results

	Lower Bound	[IMOR'18]	Greedy	Local Search
Approximation	$\Omega(k)^{\frac{k}{2}-o(k)}$	$\tilde{O}(k)^{\frac{k}{2}}$	$O(C^{k^2})$	$O(k^k)$
Core-set Size	$k^{O(1)}$	$\tilde{O}(k)$	k	k
Simple?		×	✓	✓

The Local Search Algorithm produces a composable core-set of size k with approximation factor $O(k)^{2k}$.

This Talk

The Local Search Algorithm produces a composable core-set of size k with approximation factor $O(k)^k$ for the volume maximization problem.

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The Local Search Algorithm produces a composable core-set of size k with approximation factor $O(k)^k$ for the volume maximization problem.

In comparison to the optimal core-set algorithm

- Approximation $\mathrm{O}(k)^{\boldsymbol{k}}$ as opposed to $O(k \log k)^{\boldsymbol{k/2}}$
- Smaller Size k as opposed to $O(k \log k)$
- Simpler to implement (similar to Greedy)
- Better performance in practice

The Local Search Algorithm

Input: a set V of n points and a parameter k

1. Start with an arbitrary subset of k points $S \subseteq V$
2. While there exists a point $p \in V \setminus S$ and $q \in S$ s.t. replacing q with p increases the volume, then swap them, i.e., $S = S \cup \{p\} \setminus \{q\}$

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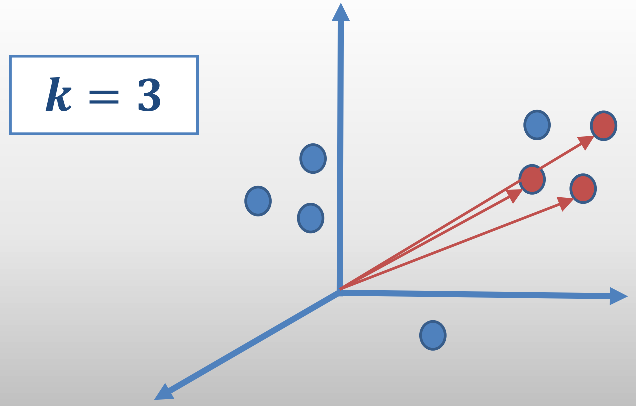

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Input: a set V of n points and a parameter k

1. Start with an arbitrary subset of k points $S \subseteq V$

2. While there exists a point $p \in V \setminus S$ and $q \in S$ s.t. replacing q with p increases the volume, then swap them, i.e., $S = S \cup \{p\} \setminus \{q\}$

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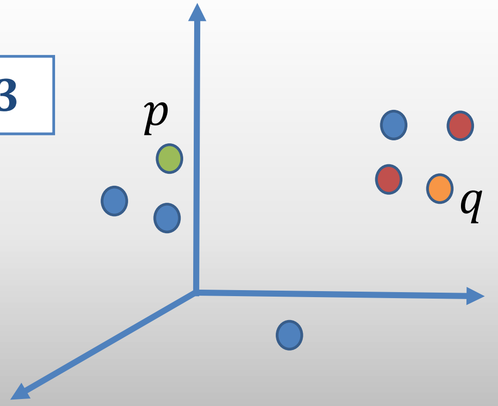

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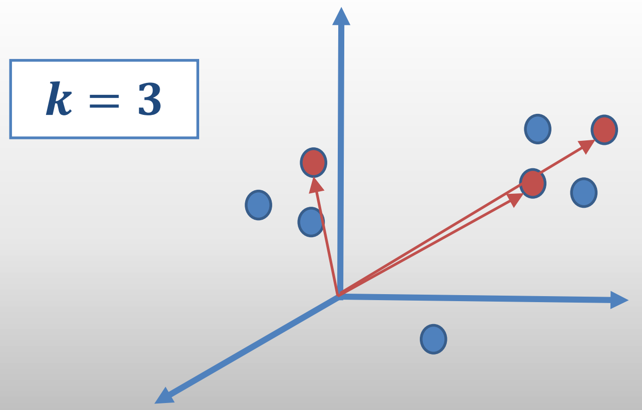

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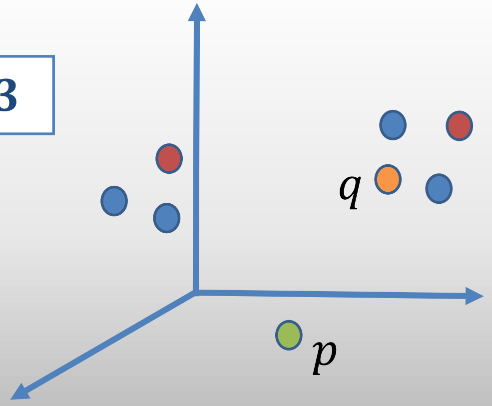

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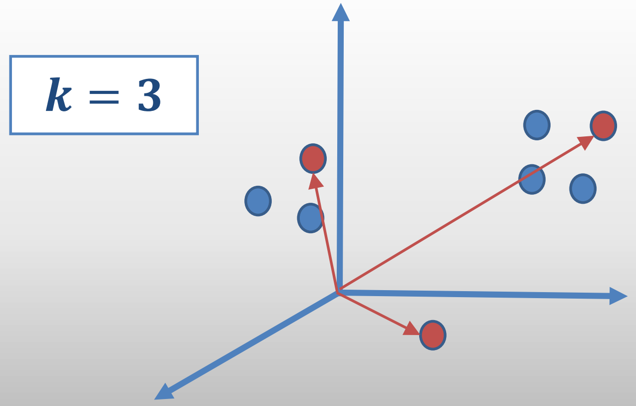

To bound the run time

Input: a set V of n points a

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Start with a crude approximation (Greedy algorithm)

If it increases by at least a factor of $(1 + \epsilon)$

Checking the condition

Input: a set V of n points and a parameter k

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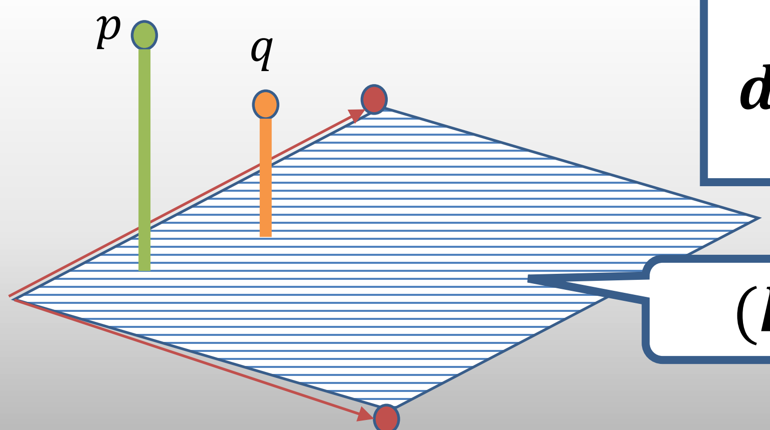
$$\boldsymbol{dist(p, H_{S\setminus\{q\}}) > dist(q, H_{S\setminus\{q\}})}$$



$(k-1)$-dimensional Subspace

Main Lemma [informal]:

Local Search preserves maximum distance to “all” subspaces of dimension $k - 1$

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- $\boldsymbol{V}$ is the point set
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For any $(k-1)$-dimensional subspace G, the maximum distance of the point set to G is approximately preserved

$$\max_{q \in S} dist(q, G) \geq \frac{1}{2k} \cdot \max_{p \in V} dist(p, G)$$

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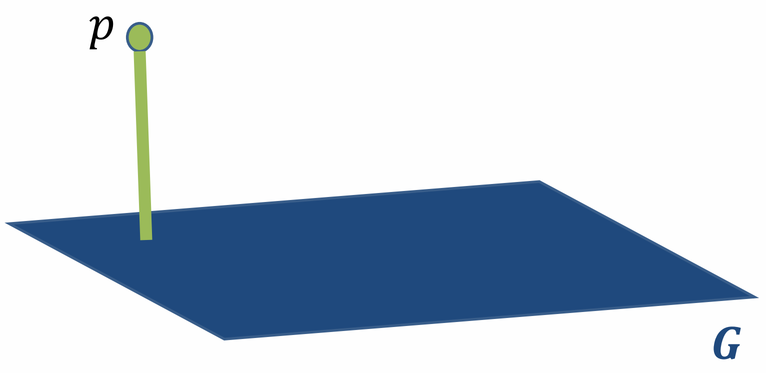

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- Let $p \in V$ be a point
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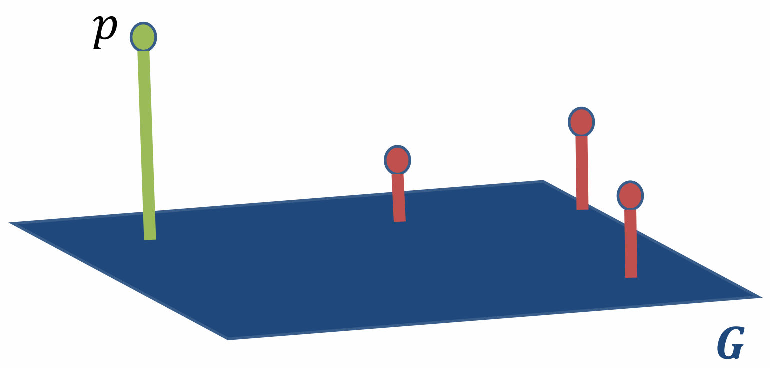
- Assume for any $q \in S$, $d(q, G) \leq x$



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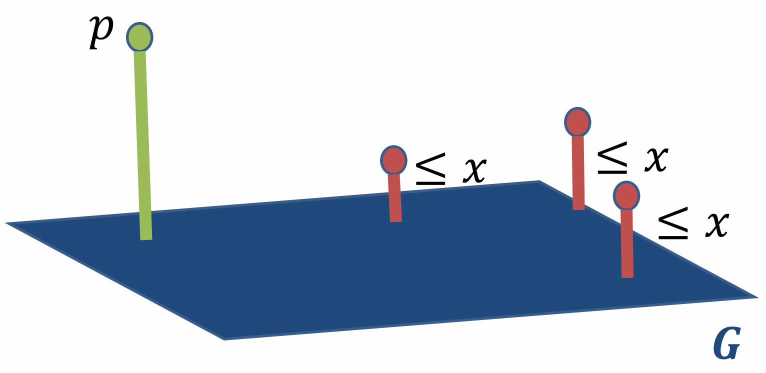

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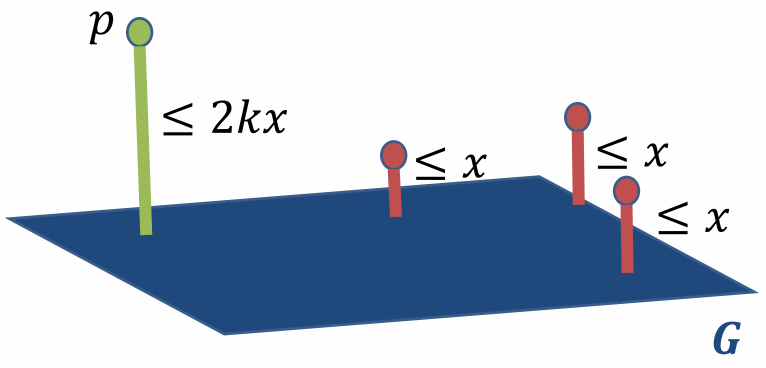
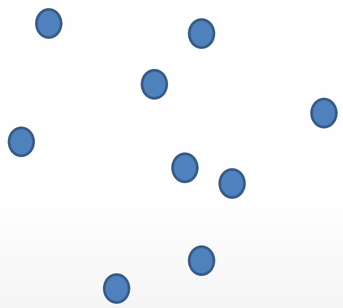


Main Theorem

Local Search produces a core-set for volume maximization

Let $\boldsymbol{V} = \bigcup_i V_i$ be the union of the point sets



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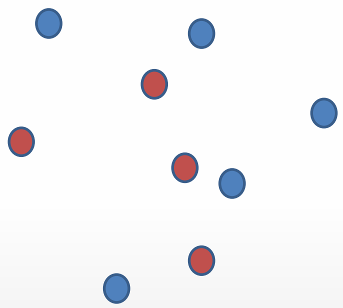

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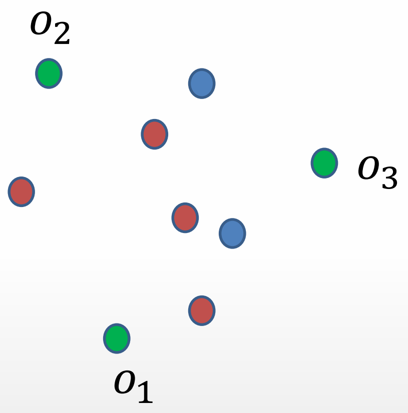

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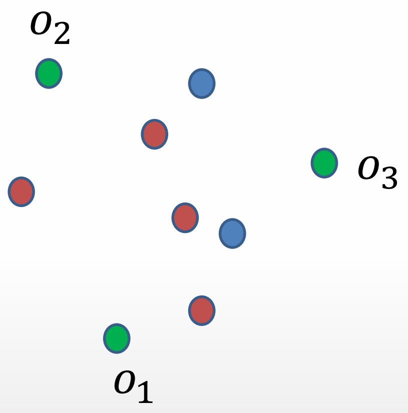

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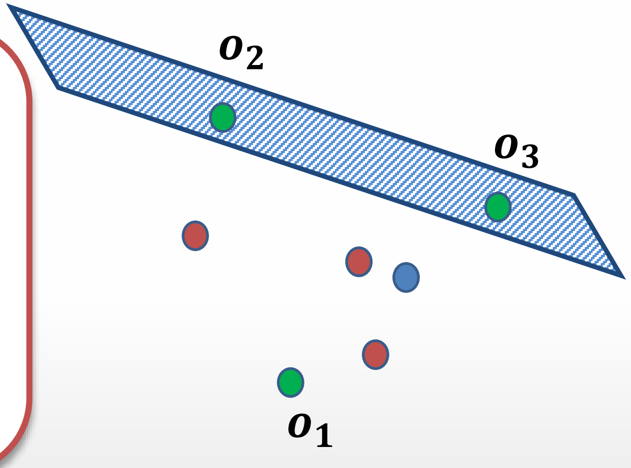

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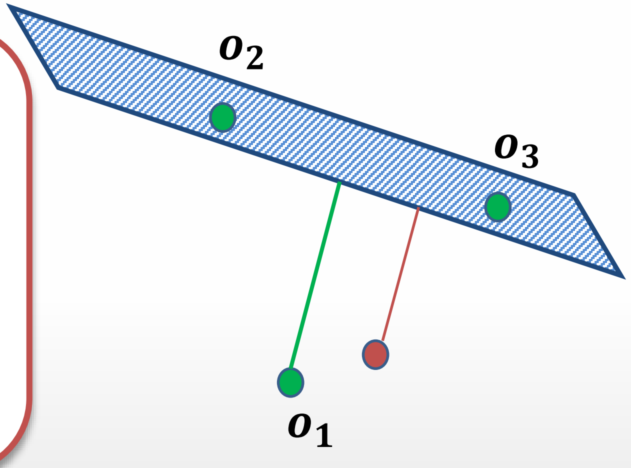

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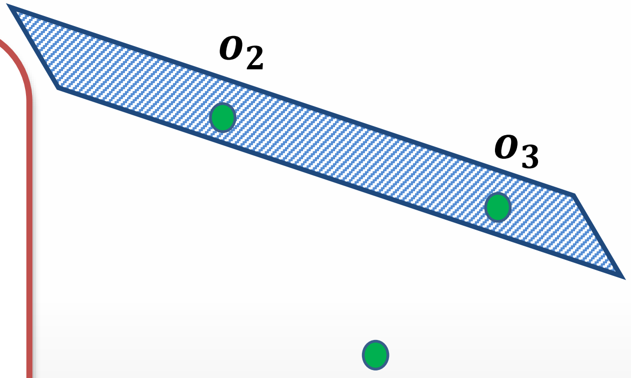

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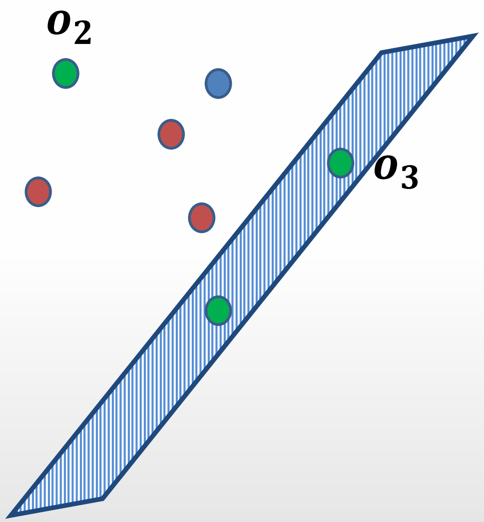

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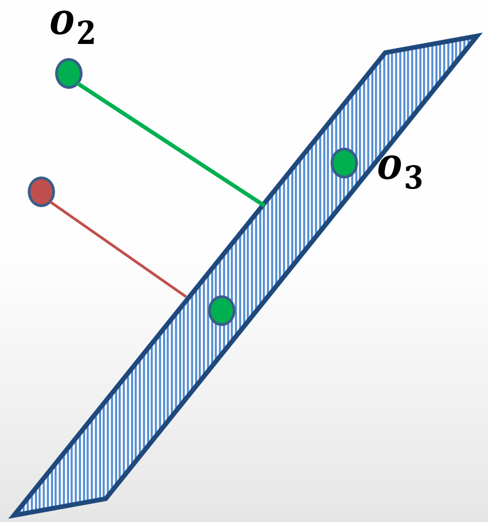

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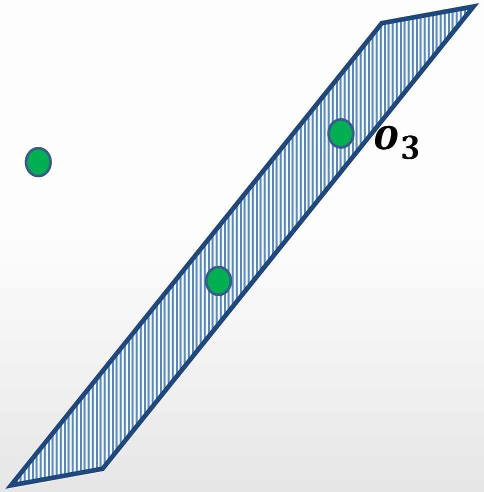

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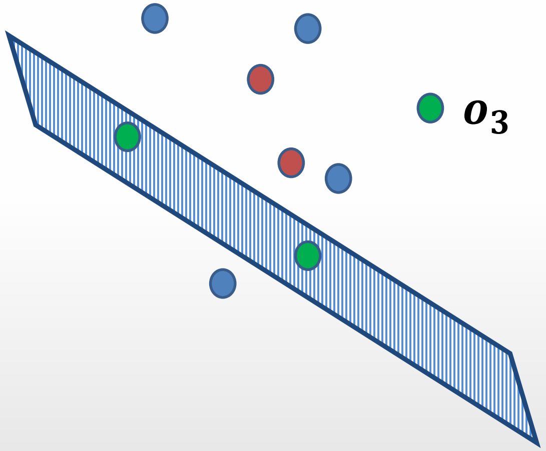

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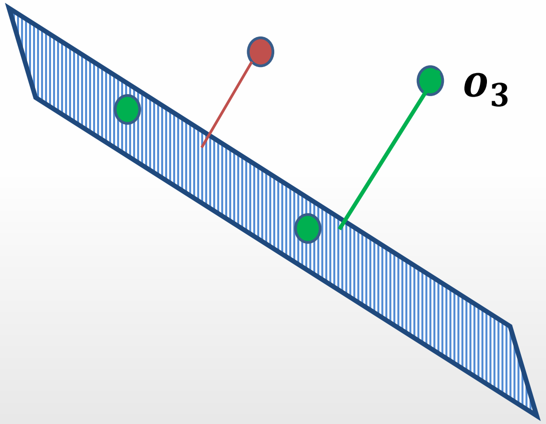

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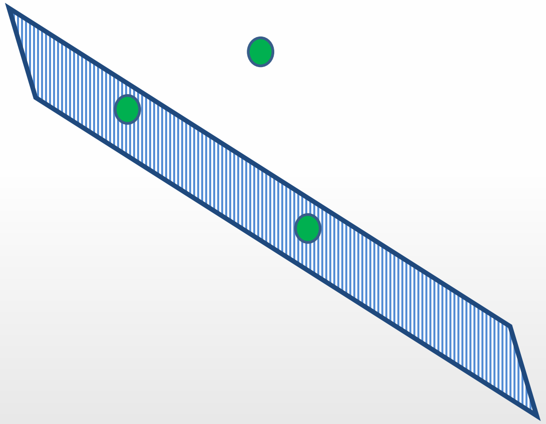

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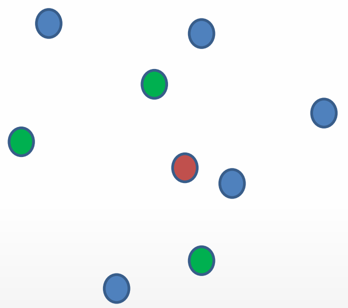

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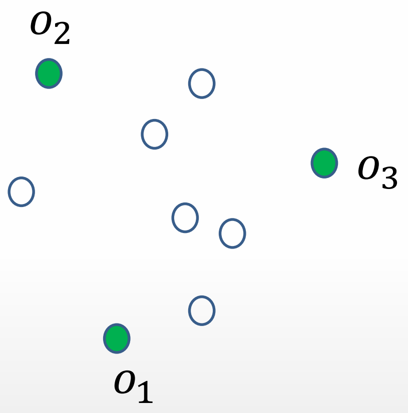

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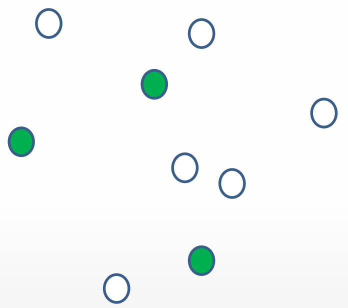

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Since local search preserve maximum distances to subspaces

➢ Lose a factor of at most $2k$ at each iteration

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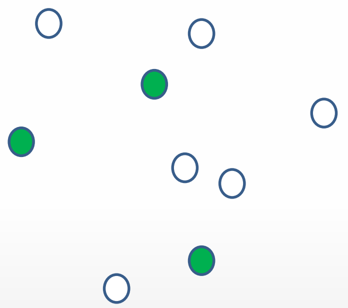


- Lose a factor of at most $2k$ at each iteration
- Total approximation factor $(2k)^k$

Empirical Results

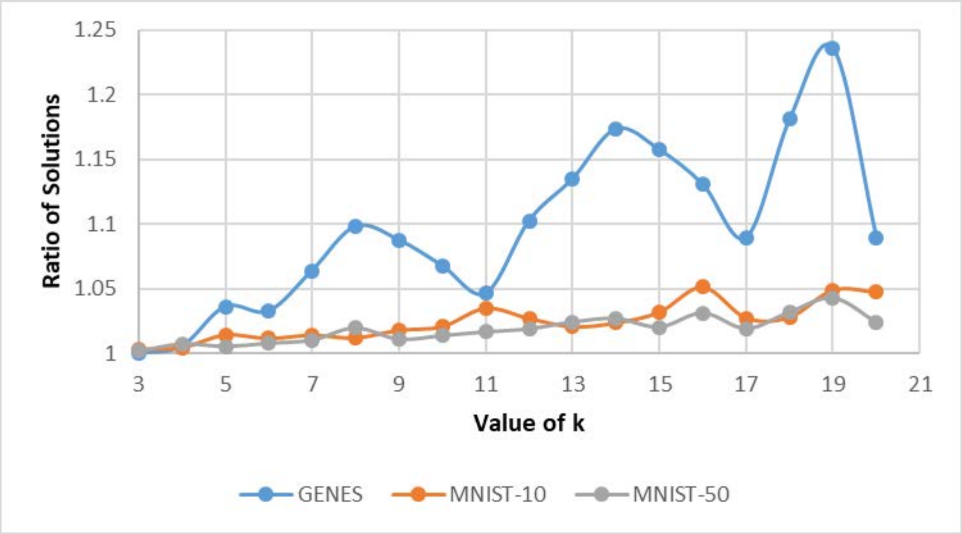
Data Set

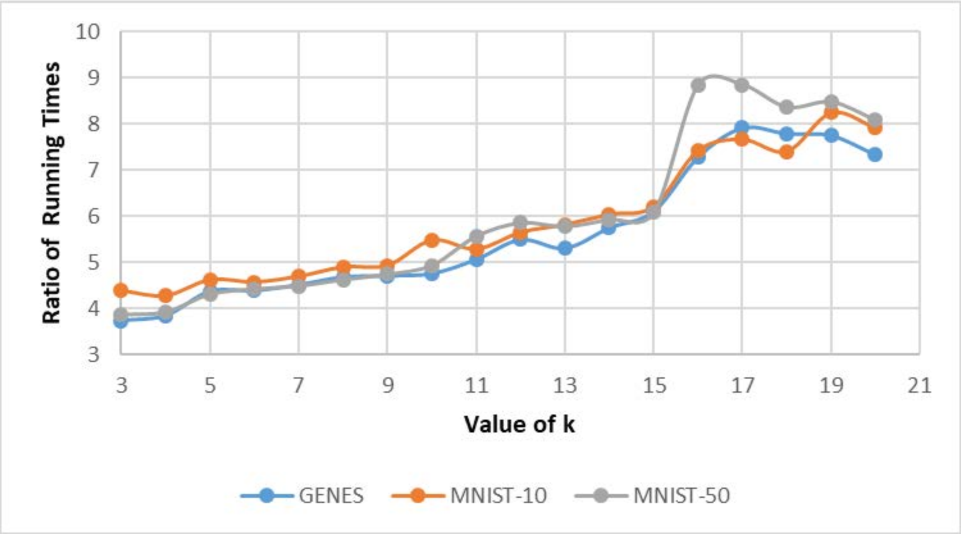
- MNIST, with number of parts = 10
- MNIST, with number of parts = 50
- GENES, with number of parts = 10

Process

- Partition the data set randomly into parts
- Compute a core-set using one of the algorithms: **Greedy, Local Search, LP-Based algorithm of [IMOR'18]**
- Use greedy on the union of the coresets

Local Search vs Greedy





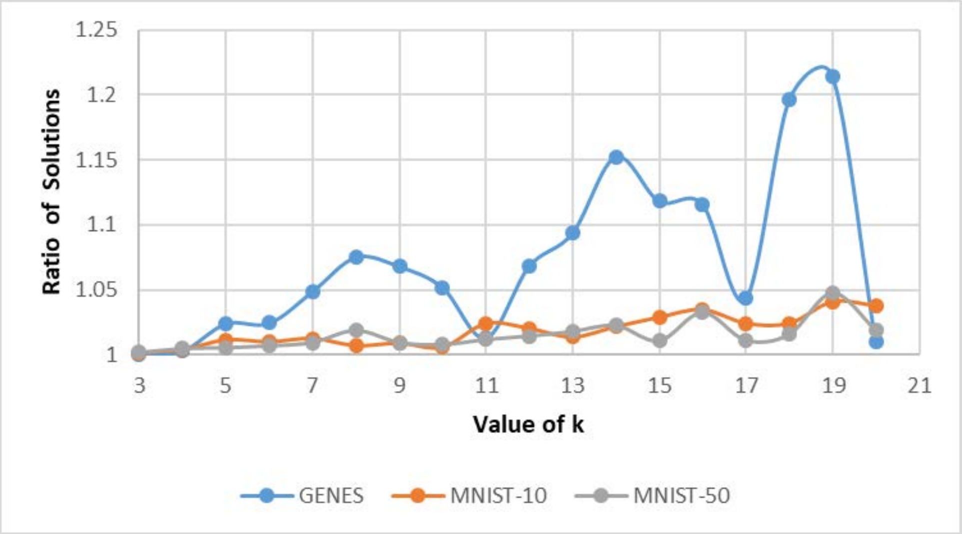
Improvement of the solution of Local Search over Greedy

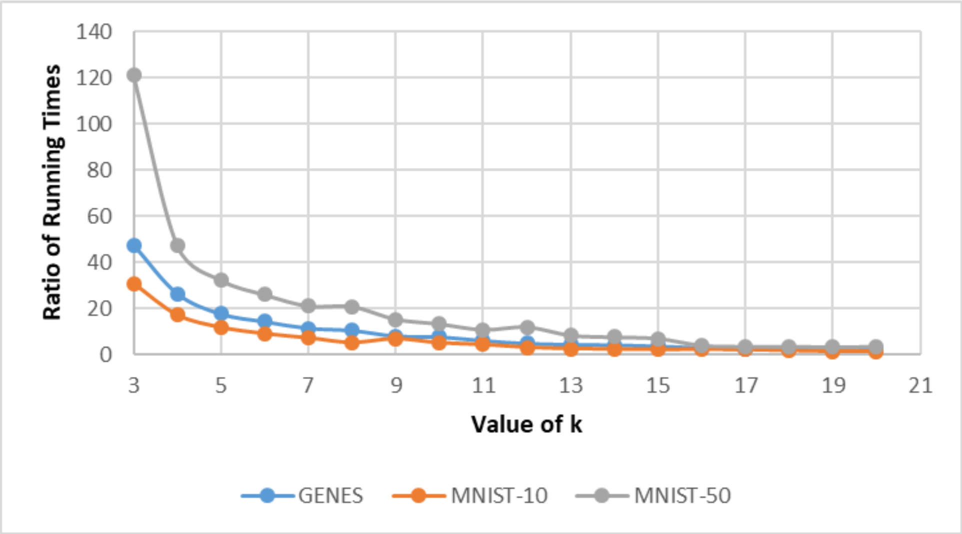
- On average, 1.2%, 2.5%, and 9.6% improvement
- Some cases up to 58% improvement

Ratio of runtime of Local Search over Greedy

- On average, 6 times slower

Local Search vs. LP-based Algorithm of [IMOR'18]





Improvement of the solution of Local Search over [IMOR'18]

- On average, 1.4%, 1.8%, and 7.3% improvement
- Some cases up to 63% improvement

Ratio of runtime of Local Search over [IMOR'18]

- For lower values of k, Local Search is up to 50 times faster.

Summary

- Volume/Determinant Maximization Problem
- Notion of composable core-sets
- Algorithms that find composable core-sets for volume/determinant maximization

	[IMOR'18]	Greedy	Local Search
Approximation	$O(k \log k)^{k/2}$	$O(C^{k^2})$	$O(k^k)$
Core-set Size	$O(k \log k)$	k	k
Simple?	×	✓	✓
Empirical Approximation			**Performs Best**
Empirical Runtime	**Slowest**	**Fastest**	**4 times slower than Greedy.**

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Core-set Size	$O(k \log k)$	k	k
Simple?	×	✓	✓
Empirical Approximation			**Performs Best**
Empirical Runtime	**Slowest**	**Fastest**	**4 times slower than Greedy.**

Conclusion

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- Notion of composable core-sets
- Algorithms that find composable core-sets for volume/determinant maximization

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THANK YOU!

Poster #81

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