Adaptive Regret of Convex and Smooth Functions

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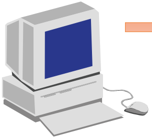
The 36th International Conference on Machine Learning (ICML 2019)

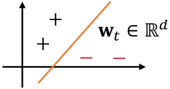
Online Learning

- Online Convex Optimization [Zinkevich, 2003]

for $t = 1, 2, \ldots, T$ **do**
    Learner picks a decision $\mathbf{w}_t \in \mathcal{W}$
    Adversary chooses a function $f_t(\cdot) : \mathcal{W} \mapsto \mathbb{R}$
    Learner suffers loss $f_t(\mathbf{w}_t)$ and updates $\mathbf{w}_t$
end for



A classifier



An example $(\mathbf{x}_t, y_t) \in \mathbb{R}^d \times \{\pm 1\}$

A loss $f_t(\mathbf{w}) = \max(1 - y_t \mathbf{w}^\top \mathbf{x}_t, 0)$

Learner

Adversary

- Cumulative Loss

$$\text{Cumulative Loss} = \sum_{t=1}^{T} f_t(\mathbf{w}_t)$$

Performance Measure

- Regret

$$\text{Regret} = \underbrace{\sum_{t=1}^{T} f_t(\mathbf{w}_t)}_{\text{Cumulative Loss of Online Learner}} - \underbrace{\min_{\mathbf{w} \in \mathcal{W}} \sum_{t=1}^{T} f_t(\mathbf{w})}_{\text{Minimal Loss of Offline Learner}}$$

Performance Measure

- Regret

$$\text{Regret} = \underbrace{\sum_{t=1}^{T} f_t(\mathbf{w}_t)}_{\text{Cumulative Loss of Online Learner}} - \underbrace{\min_{\mathbf{w}\in\mathcal{W}} \sum_{t=1}^{T} f_t(\mathbf{w})}_{\text{Minimal Loss of Offline Learner}}$$

- Convex Functions [Zinkevich, 2003]
  - Online Gradient Descent (OGD)

$$\text{Regret} = O\left(\sqrt{T}\right)$$

- Convex and Smooth Functions [Srebro et al., 2010]
  - OGD with prior knowledge

$$\text{Regret} = O\left(1 + \sqrt{F_*}\right)$$

where $F_* = \min_{\mathbf{w}\in\mathcal{W}} \sum_{t=1}^{T} f_t(\mathbf{w})$

- Exp-concave Functions [Hazan et al., 2007]
- Strongly Convex Functions [Hazan et al., 2007]

Learning in Changing Environments

- Regret $\rightarrow$ Static Regret
  - $$\begin{aligned}\text{Regret} &= \sum_{t=1}^{T} f_t(\mathbf{w}_t) - \min_{\mathbf{w}\in\mathcal{W}} \sum_{t=1}^{T} f_t(\mathbf{w}) \\ &= \sum_{t=1}^{T} f_t(\mathbf{w}_t) - \sum_{t=1}^{T} f_t(\mathbf{w}_*)\end{aligned}$$

    where $\mathbf{w}_* \in \operatorname{argmin}_{\mathbf{w}\in\mathcal{W}} \sum_{t=1}^{T} f_t(\mathbf{w})$
  - $\mathbf{w}_*$ is reasonably good during T rounds
- Changing Environments
  - Different decisions will be good in different periods
  - E.g., recommendation, stock market

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Adaptive Regret

The Basic Idea

Minimize the regret over *every* interval $[r, s]$

$$\text{Regret}\big([r, s]\big) = \sum_{t=r}^{s} f_t(\mathbf{w}_t) - \min_{\mathbf{w}\in\mathcal{W}} \sum_{t=r}^{s} f_t(\mathbf{w})$$

- Weakly Adaptive Regret [Hazan and Seshadhri, 2007]

$$\text{WA-Regret}(T) = \max_{[r,s]\subseteq[T]} \text{Regret}\big([r, s]\big)$$

  - The maximal regret over all intervals

- Strongly Adaptive Regret [Daniely et al., 2015]

$$\text{SA-Regret}(T, \tau) = \max_{[s,s+\tau-1]\subseteq[T]} \text{Regret}\big([s, s+\tau-1]\big)$$

  - The maximal regret over all intervals of length τ

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State-of-the-Art

- Convex Functions [Jun et al., 2017]

$$\text{Regret}\left([r, s]\right) = O\left(\sqrt{(s-r)\log s}\right)$$

$$\Rightarrow \text{SA-Regret}(T, \tau) = O\left(\sqrt{\tau \log T}\right)$$

- Exp-concave Functions [Hazan and Seshadhri, 2007]
- Strongly Convex Functions [Zhang et al., 2018]

State-of-the-Art

- Convex Functions [Jun et al., 2017]

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- Exp-concave Functions [Hazan and Seshadhri, 2007]
- Strongly Convex Functions [Zhang et al., 2018]

Question

Can smoothness be exploited to boost the adaptive regret?

Our Results

- Convex and Smooth Functions

$$\text{Regret}\left([r,s]\right) = O\left(\sqrt{\left(\sum_{t=r}^{s} f_t(\mathbf{w})\right)\log s \cdot \log(s-r)}\right)$$

  - Become tighter when $\sum_{t=r}^{s} f_t(\mathbf{w})$ is small

- Convex Functions [Jun et al., 2017]

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- Convex and Smooth Functions

$$\text{Regret}([r,s]) = O\left(\sqrt{\left(\sum_{t=r}^{s} f_t(\mathbf{w})\right) \log \sum_{t=1}^{s} f_t(\mathbf{w}) \cdot \log \sum_{t=r}^{s} f_t(\mathbf{w})}\right)$$

  - Fully problem-dependent

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The Algorithm

- An Expert-algorithm
  - Scale-free online gradient descent [Orabona and Pál, 2018]
  - Can exploit smoothness automatically

- A Set of Intervals
  - Compact geometric covering intervals [Daniely et al., 2015]

t	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	$\cdots$
$\mathcal{C}_0$	[ ]		[ ]		[ ]		[ ]		[ ]		[ ]		[ ]		[ ]		[ ]		$\cdots$
$\mathcal{C}_1$		[	]			[	]			[	]			[	]			[	$\cdots$
$\mathcal{C}_2$				[			]					[			]				$\cdots$
$\mathcal{C}_3$								[							]				$\cdots$
$\mathcal{C}_4$																[			$\cdots$

- A Meta-algorithm
  - AdaNormalHedge [Luo and Schapire, 2015]
  - Attain a small-loss regret and support sleeping experts

Thanks!

Welcome to Our Poster @ Pacific Ballroom #161.

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