Maximum Likelihood Estimation for Learning Populations of Parameters

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joint work with
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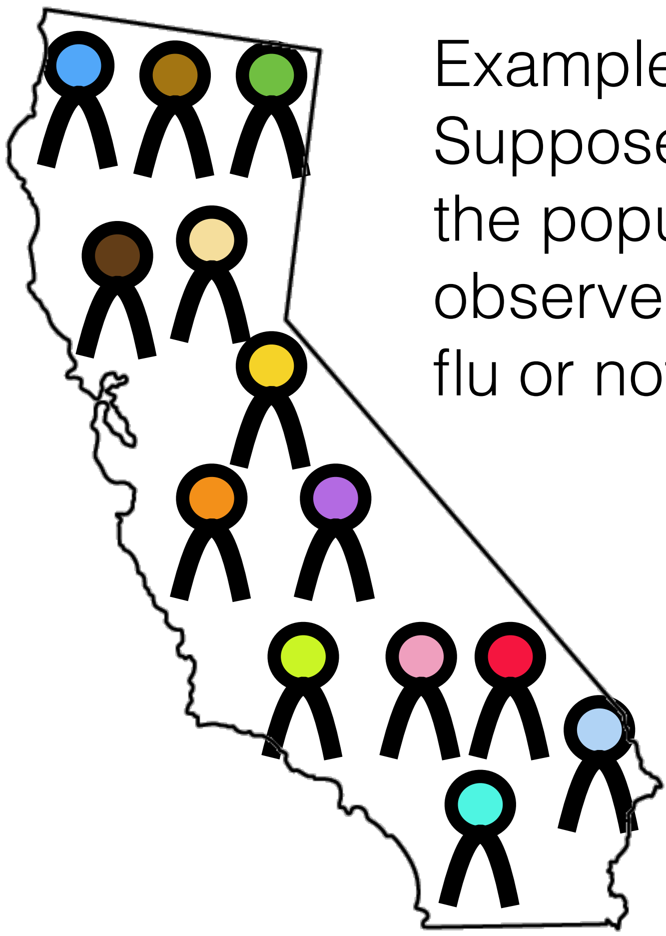
Poster #189

Motivation: Large yet Sparse Data



Example: Flu data
Suppose for a large random subset of the population in California, we observe whether a person caught the flu or not for last 5 years

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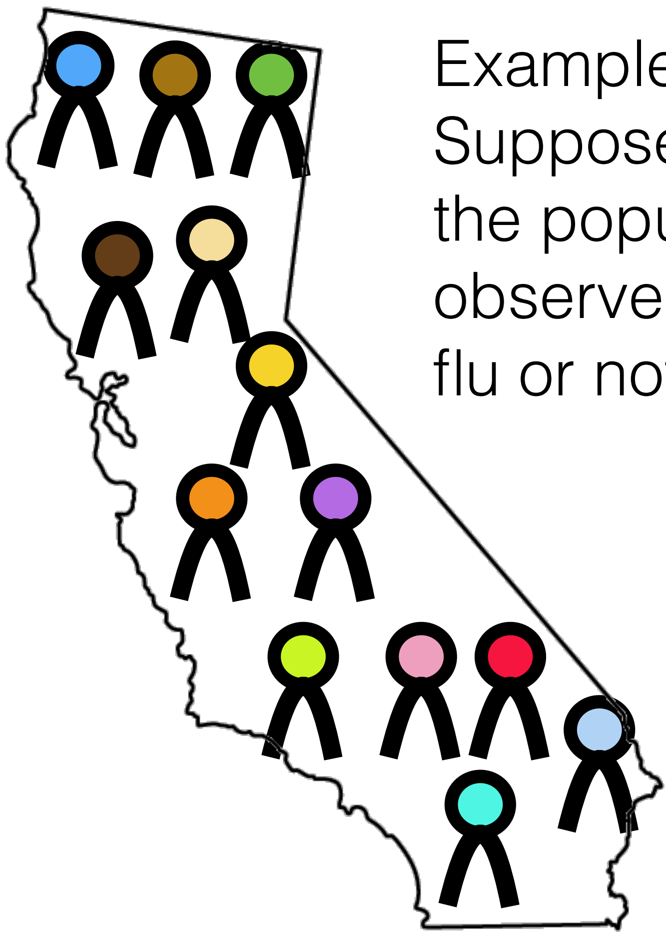

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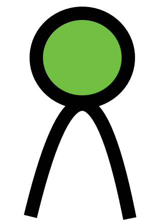

i p_i

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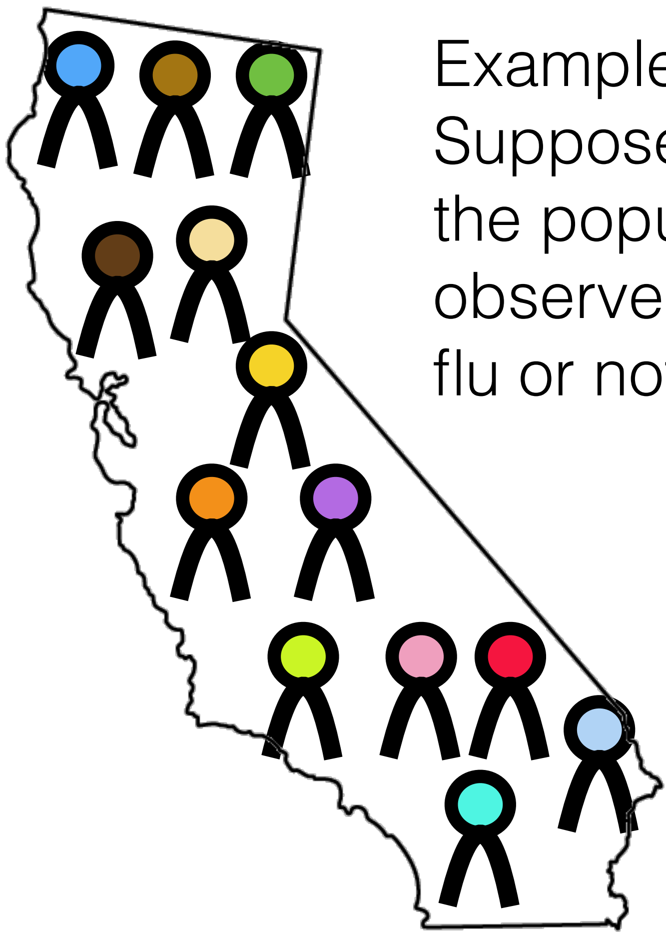

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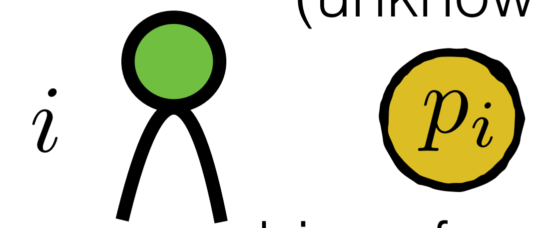

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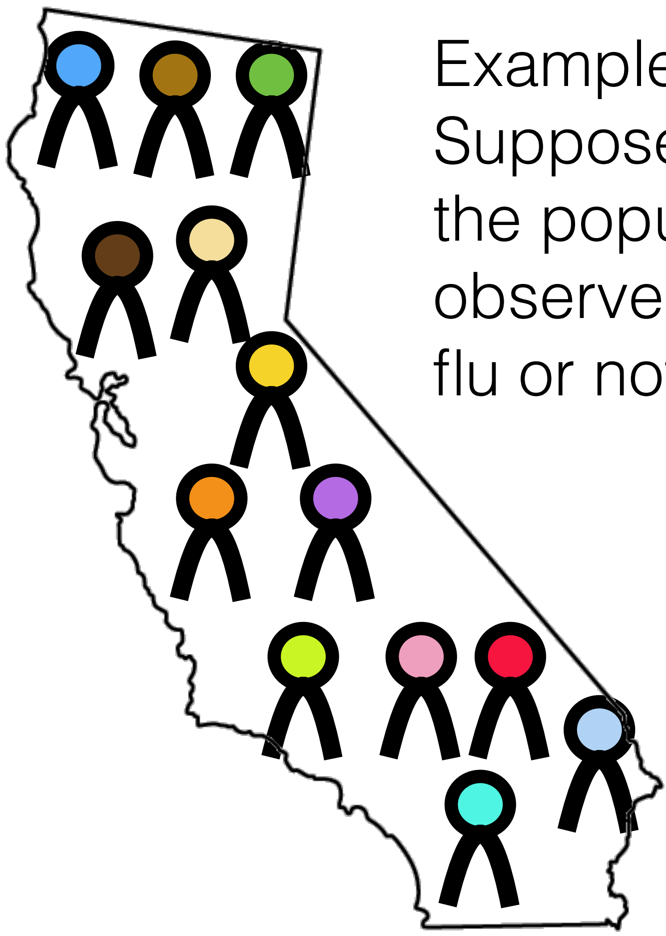

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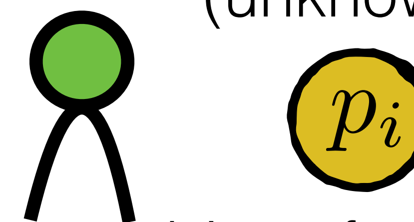

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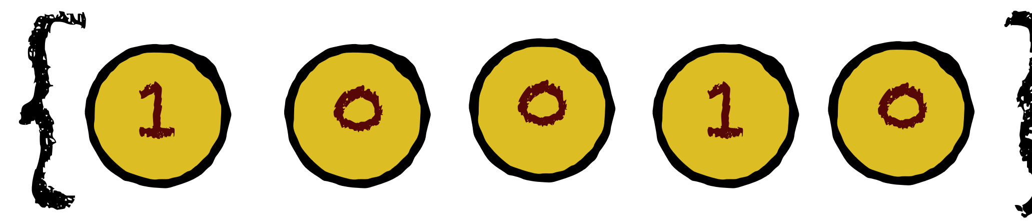

$$x_i = 2$$

$$\widehat{p}_i = \frac{x_i}{t} = 0.4 \pm 0.45$$

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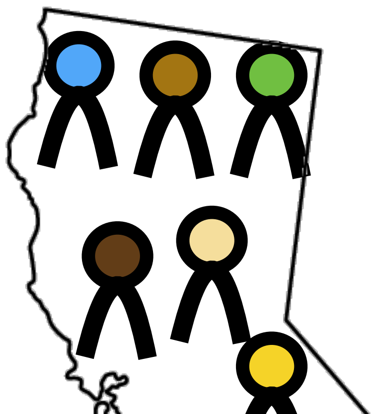

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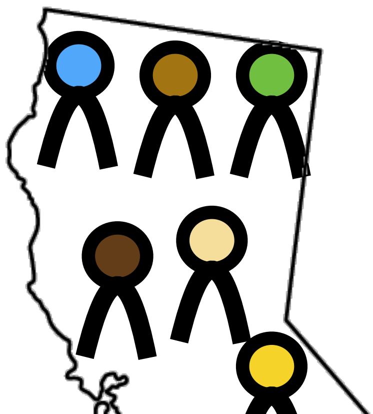

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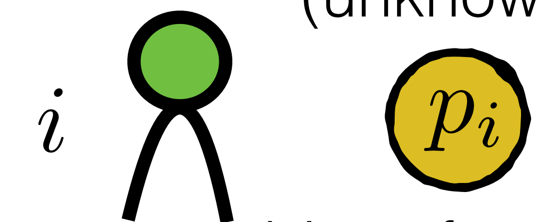

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Goal: Can we learn the distribution of the biases over the population?

- Application domains: Epidemiology, Social Sciences, Psychology, Medicine, Biology
- Population size is large, often hundreds of thousands or millions
- Number of observations per individual is limited (*sparse*) prohibiting accurate estimation of parameters of interest

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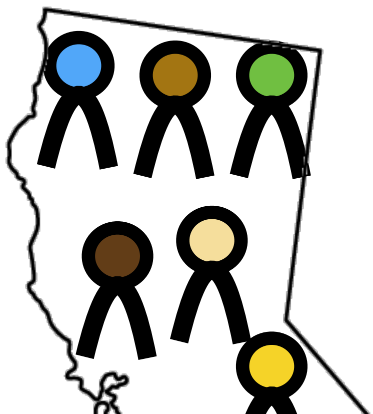

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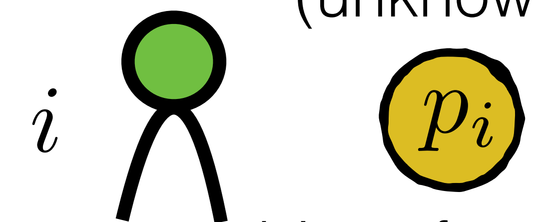

bias of coin i

Goal: Can we learn the distribution of the biases over the population?

Why?

Useful for downstream analysis:

Testing and estimating properties of the distribution

Model: Non-parametric Mixture of Binomials

Lord 1965, 1969

- N independent coins
  Each coin has its own bias drawn from $P^\star$

$i = 1, 2, ..., N$ $\quad p_i \sim P^\star$ (unknown)

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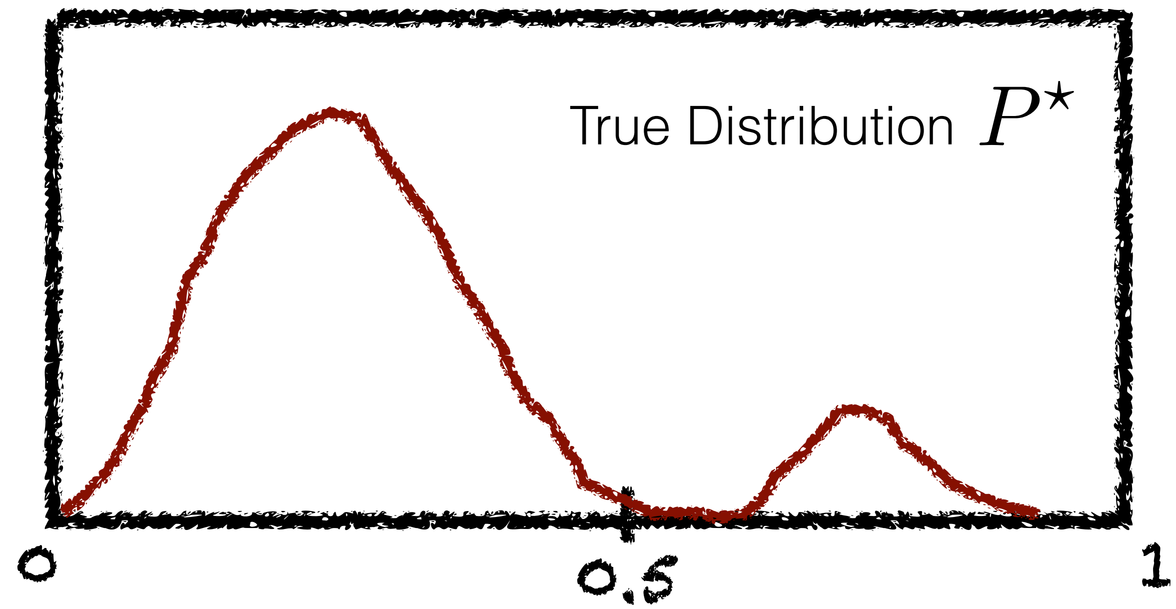

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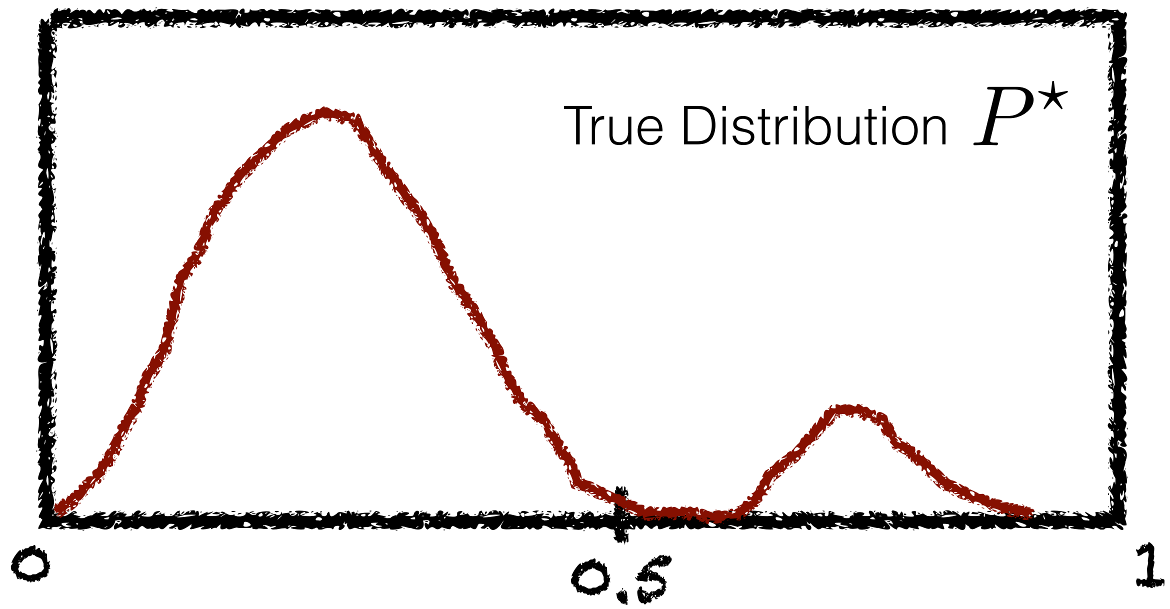

- We get to observe t tosses for every coin

  Observations: $X_i \sim \mathrm{Bin}(t, p_i) \in \{0, 1, ..., t\}$

  $t = 5$ tosses $\{1 \quad 0 \quad 0 \quad 1 \quad 0\}$ $x_i = 2$

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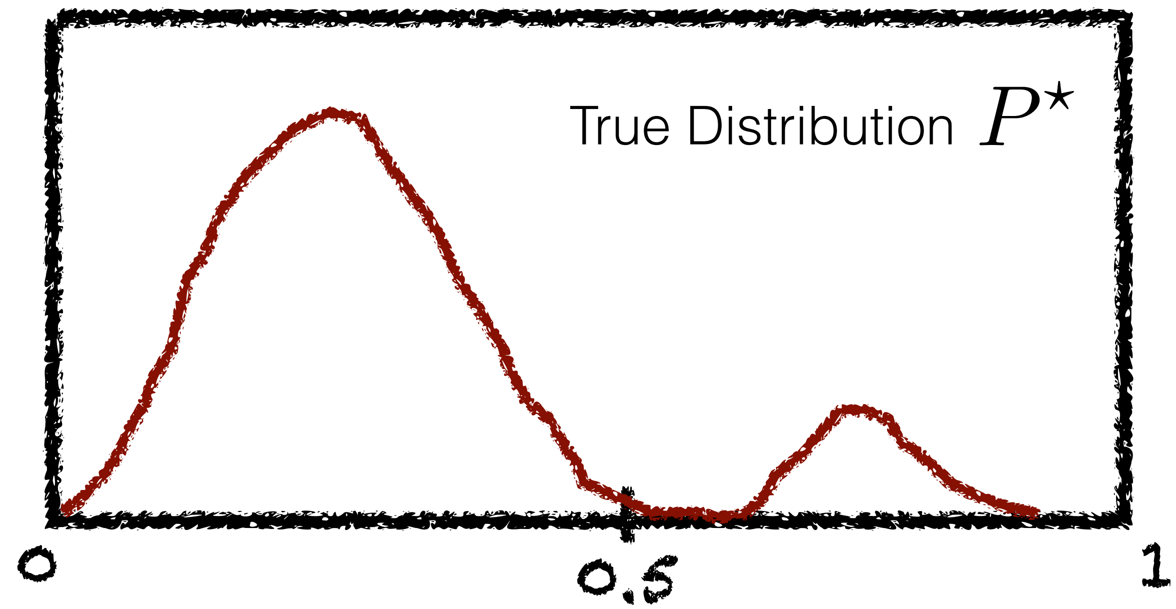

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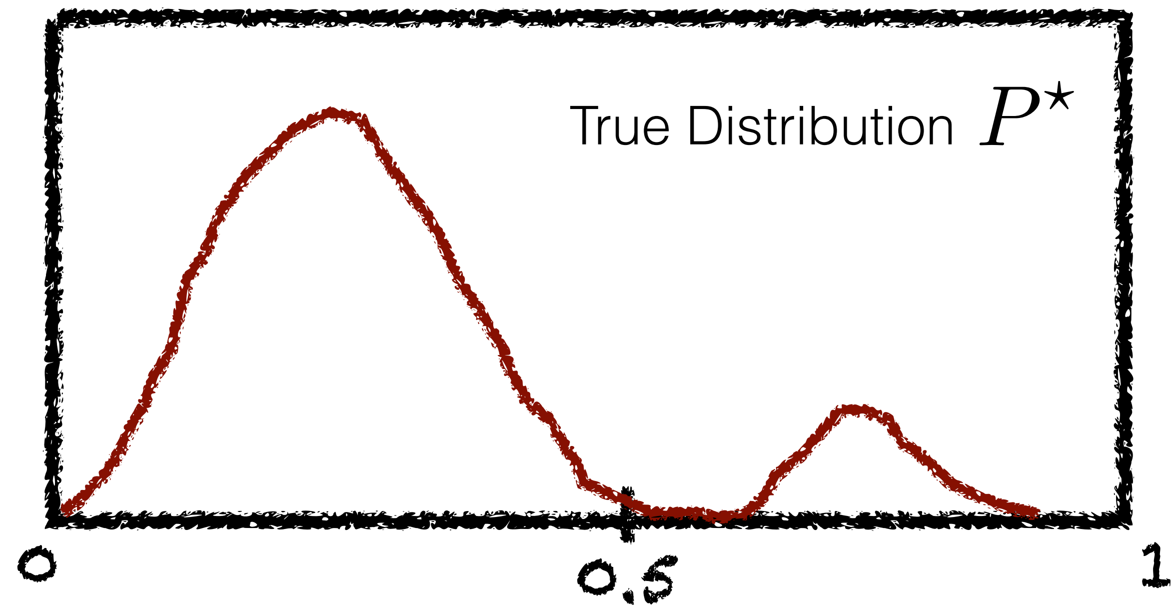

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- Wasserstein-1 distance
  (Earth Mover's Distance) $\quad W_1\left(P^\star, \hat{P}\right)$

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Learning with Sparse Observations is Non-trivial

- Empirical plug-in estimator is bad $\hat{P}_{\text{plug-in}} = \text{histogram}\left\{\frac{X_1}{t}, \ldots \frac{X_i}{t}, \ldots, \frac{X_N}{t}\right\}$

  When $t \ll N$ incurs error of $\Theta\left(\frac{1}{\sqrt{t}}\right)$

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- Many recent works on estimating symmetric properties of a discrete distribution with sparse observations

  Paninski 2003, Valiant and Valiant 2011, Jiao et. al. 2015, Orlitsky et. al. 2016, Acharya et. al. 2017 ….

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- Tian et. al 2017 proposed a moment matching based estimator which achieves optimal error of $\mathcal{O}\left(\frac{1}{t}\right)$ when $t < c\log N$

  Weakness of moment matching estimator is that it fails to obtain optimal error when $t > c \log N$ due to higher variance in larger moments

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What about Maximum Likelihood Estimator?

Weakness of moment matching estimator is that it fails to obtain optimal error when $t > c \log N$ due to higher variance in larger moments

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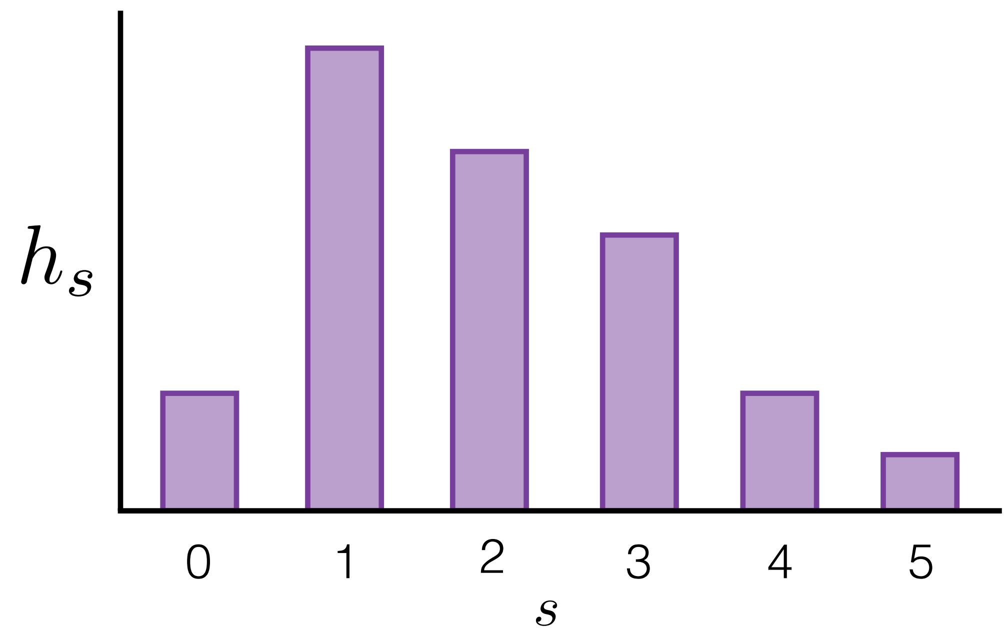
Maximum Likelihood Estimator

Sufficient statistic: Fingerprint

$$h_s = \frac{\#\text{ coins that show s heads}}{N} \qquad s = 0, 1, ..., t$$

$\mathbf{h} = [h_0, h_1, ..h_s, .., h_t]$ fingerprint vector



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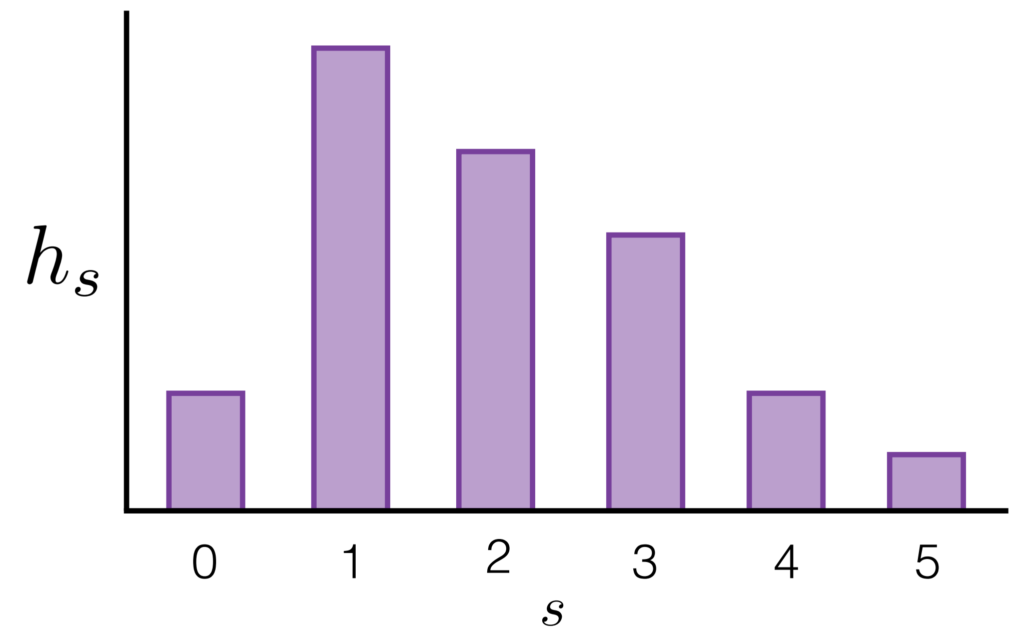

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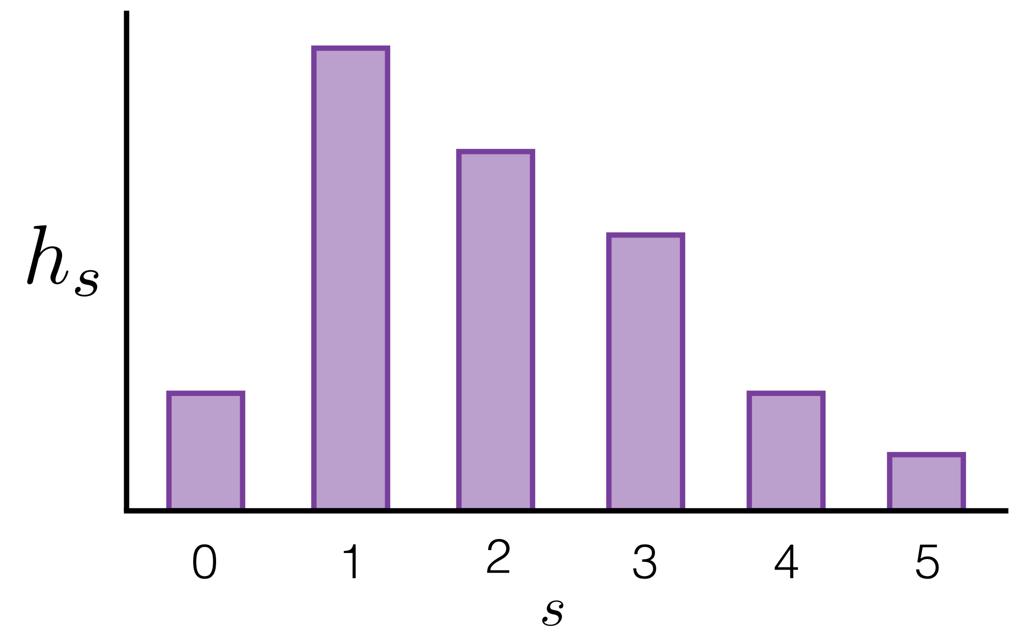

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- Convex optimization: Efficient (polynomial time)

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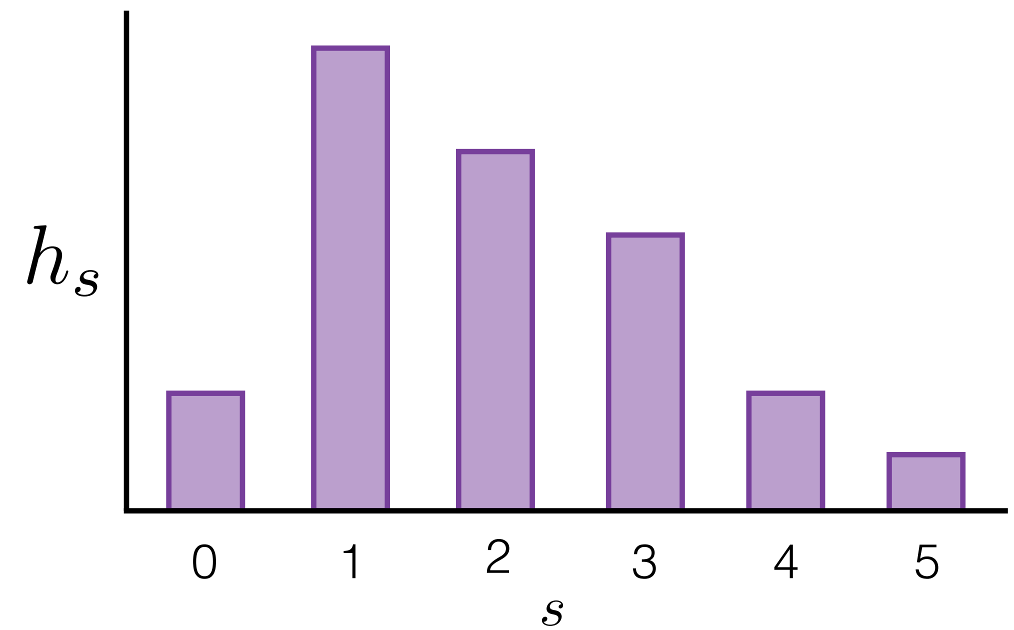

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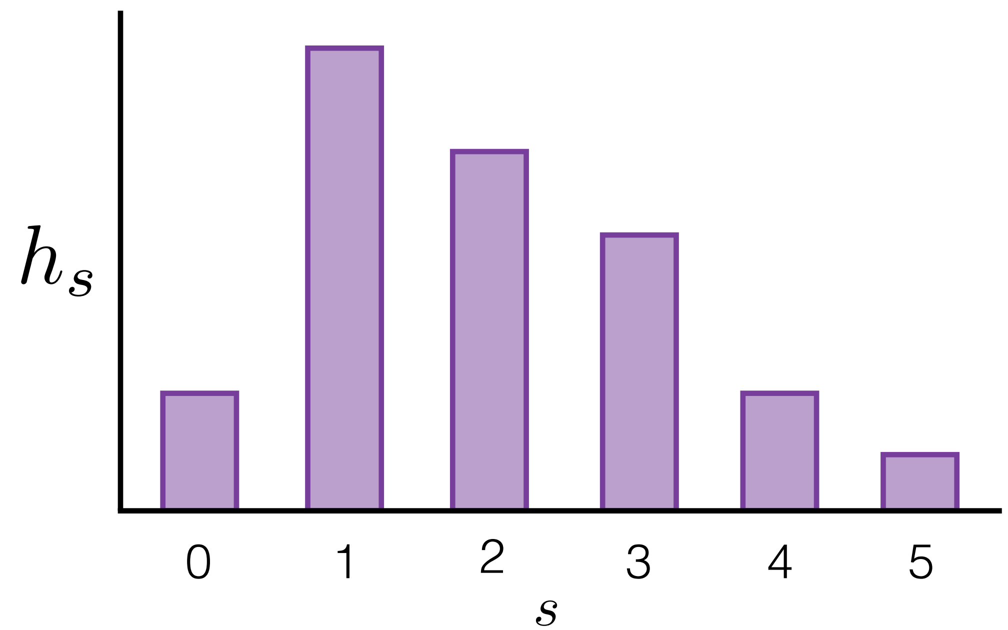

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How well does the MLE recover the distribution?

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Main Results: MLE is Minimax Optimal in Sparse Regime

Non-asymptotic guarantees

Theorem 1

The MLE achieves following error bounds: $\text{w. p. } \geq 1-\delta$

- Small Sample Regime:

$$W_1\left(P^\star, \hat{P}_{\text{mle}}\right) = \mathcal{O}_\delta\left(\frac{1}{t}\right) \quad \text{when} \quad t < c\log N$$

- Medium Sample Regime:

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Sparse Regime
$t \ll N$

Theorem 2

- Matching Minimax Lower Bounds

$$\inf_f \sup_P \; \mathrm{E}\left[W_1(P, f(\mathbf{X}))\right] > \Omega\left(\frac{1}{t}\right) \vee \Omega\left(\frac{1}{\sqrt{t\ \log N}}\right)$$

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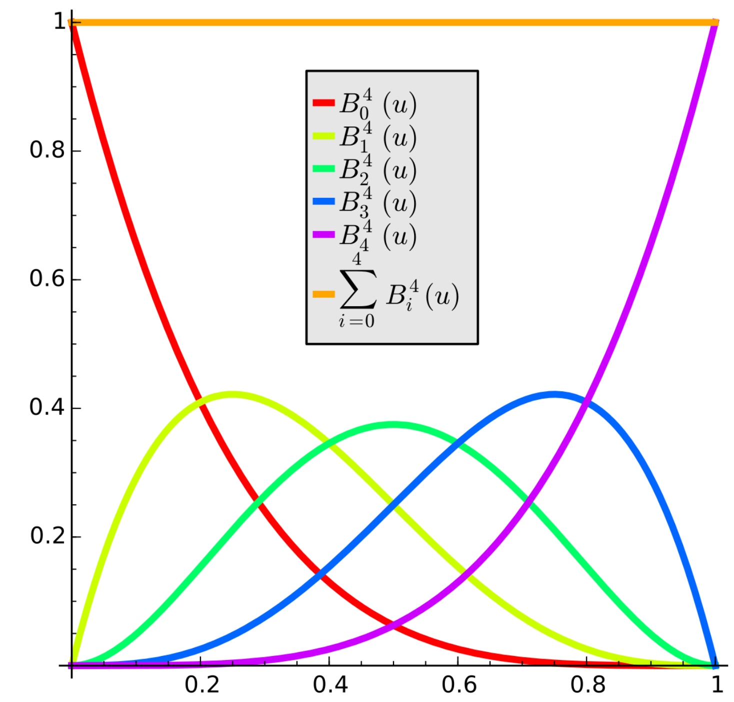
Novel Proof: Polynomial Approximations

New bounds on coefficients of Bernstein polynomials approximating Lipschitz-1 functions on [0, 1]

$$\widehat{f}_t(x) = \sum_{j=0}^{t} \boxed{b_j} \binom{t}{j} x^j (1-x)^{t-j}$$



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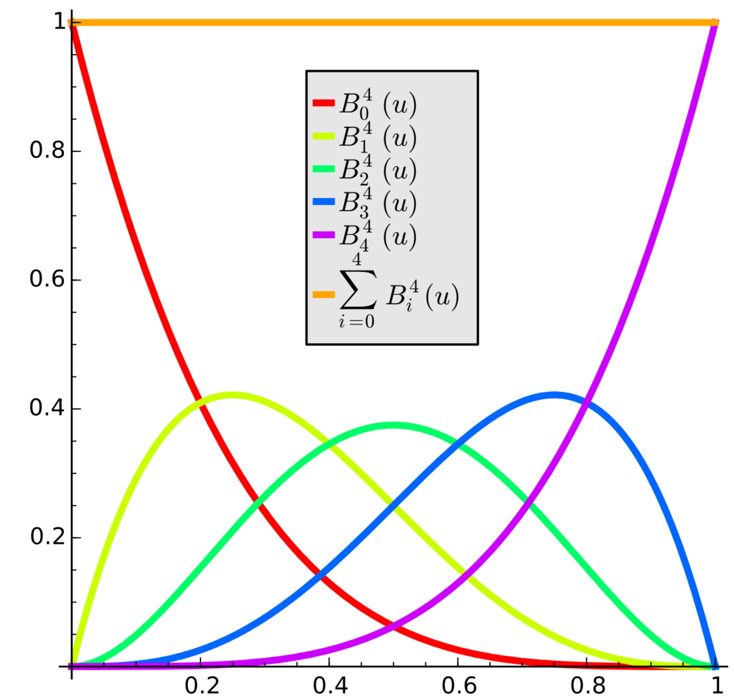
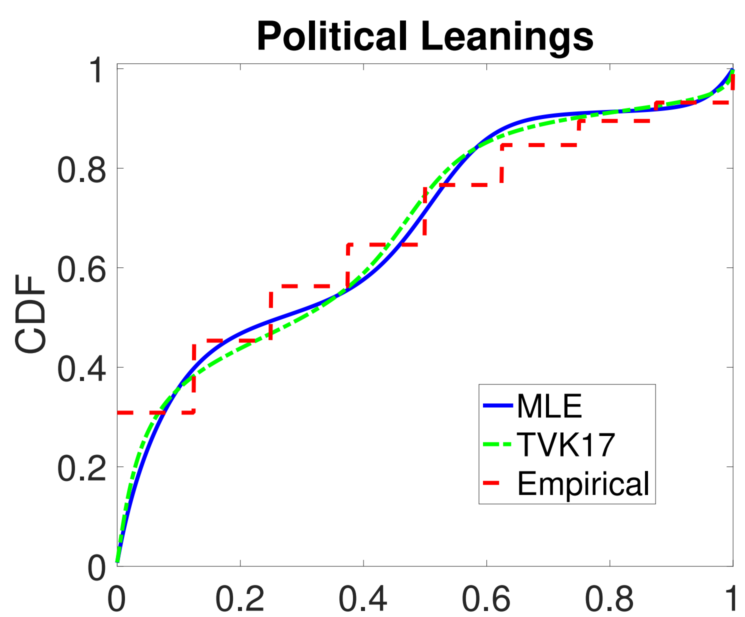
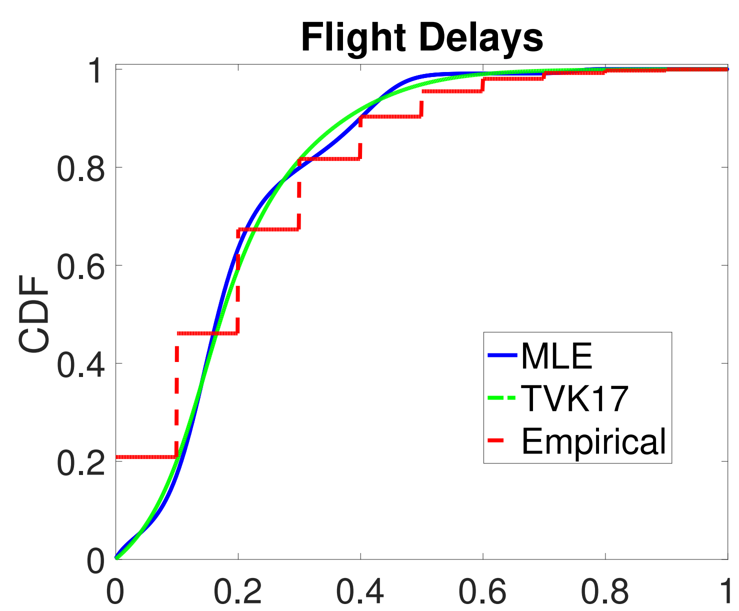


Bernstein polynomials

Performance on Real Data





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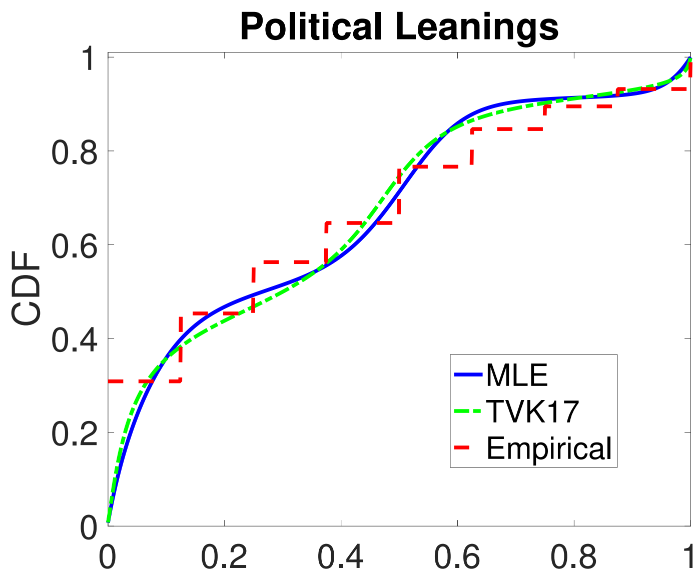
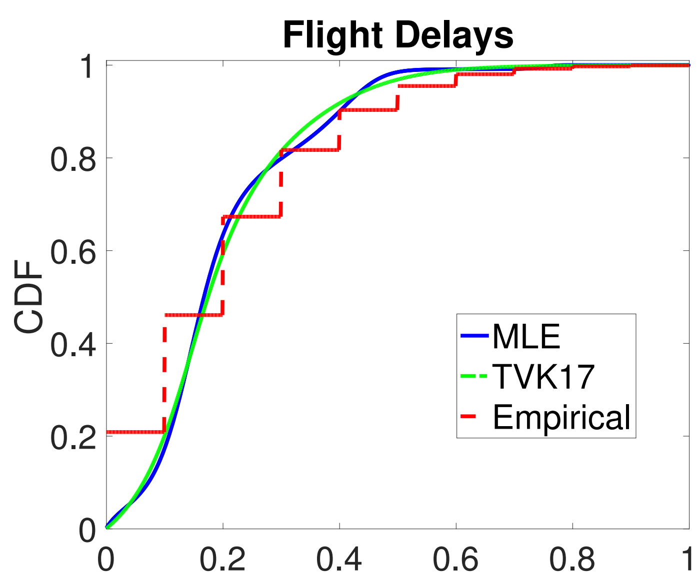
Summary

Learning distribution of parameters over a population with sparse observations per individual

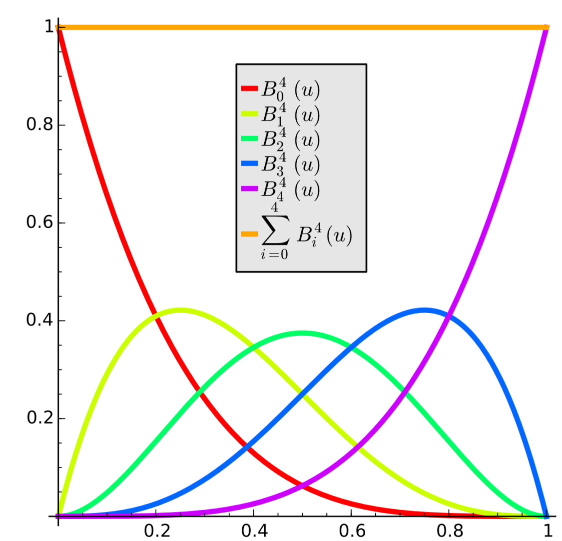
Performance on Real Data





MLE is Minimax Optimal even with sparse observations!

Novel proof: new bounds on coefficients of Bernstein polynomials approximating Lipschitz-1 functions



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