Online Learning with Kernel Losses

Aldo Pacchiano

UC Berkeley

Joint work with Niladri Chatterji and Peter Bartlett

Talk Overview

- Intro to Online Learning
- Linear Bandits
- **Kernel Bandits**

Online Learning

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Learner

$$t = 1, \cdots, n$$

Adversary

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The learner's objective is to minimize Regret

Full information vs Bandit feedback

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Full Information:

Learner gets to sees all of

$$\ell_t(\cdot)$$

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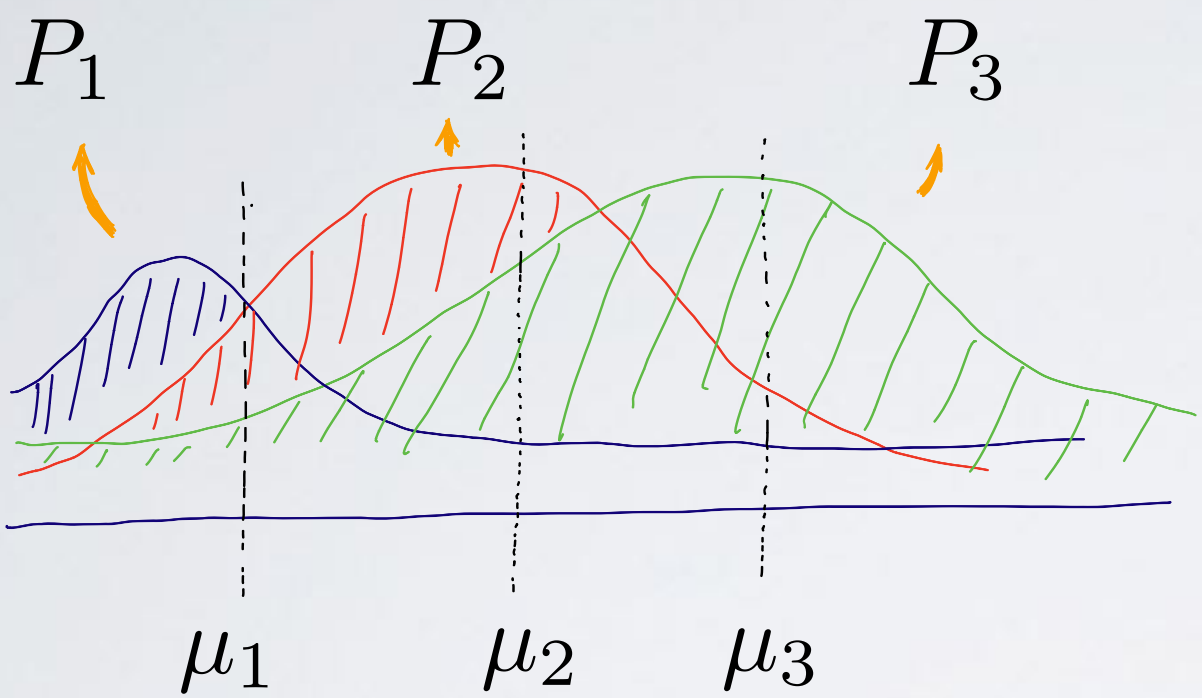
Multi Armed Bandits



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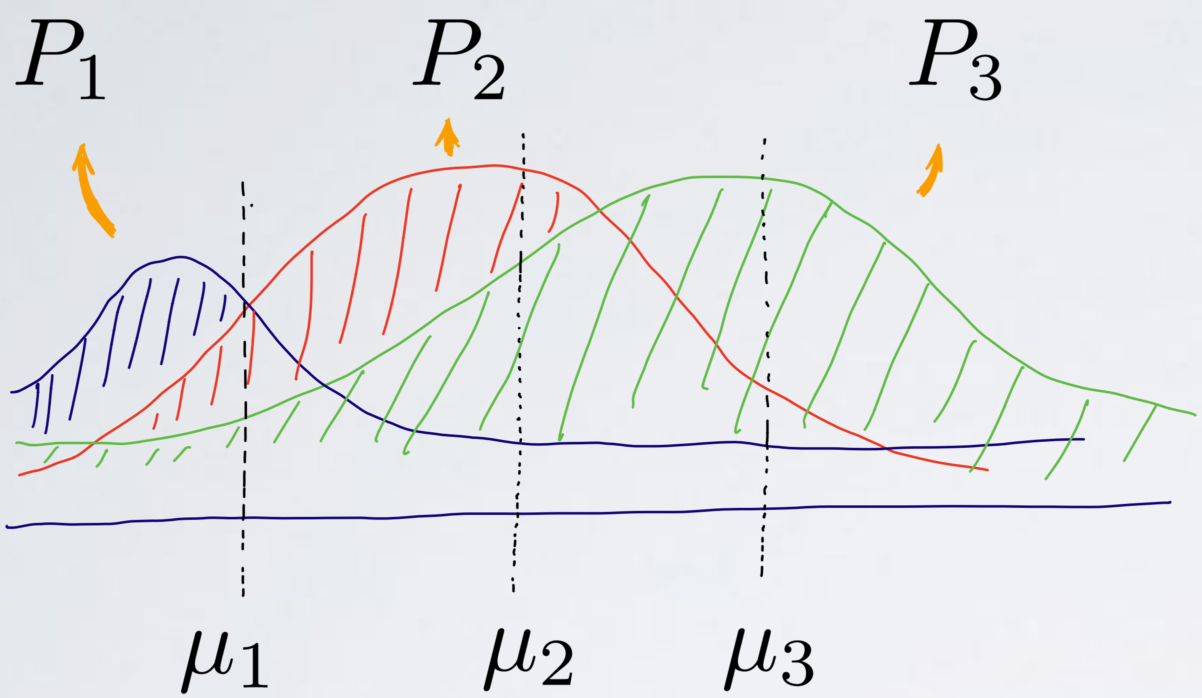

Learner chooses

$$a_t \in \{1, \cdots, K\}$$

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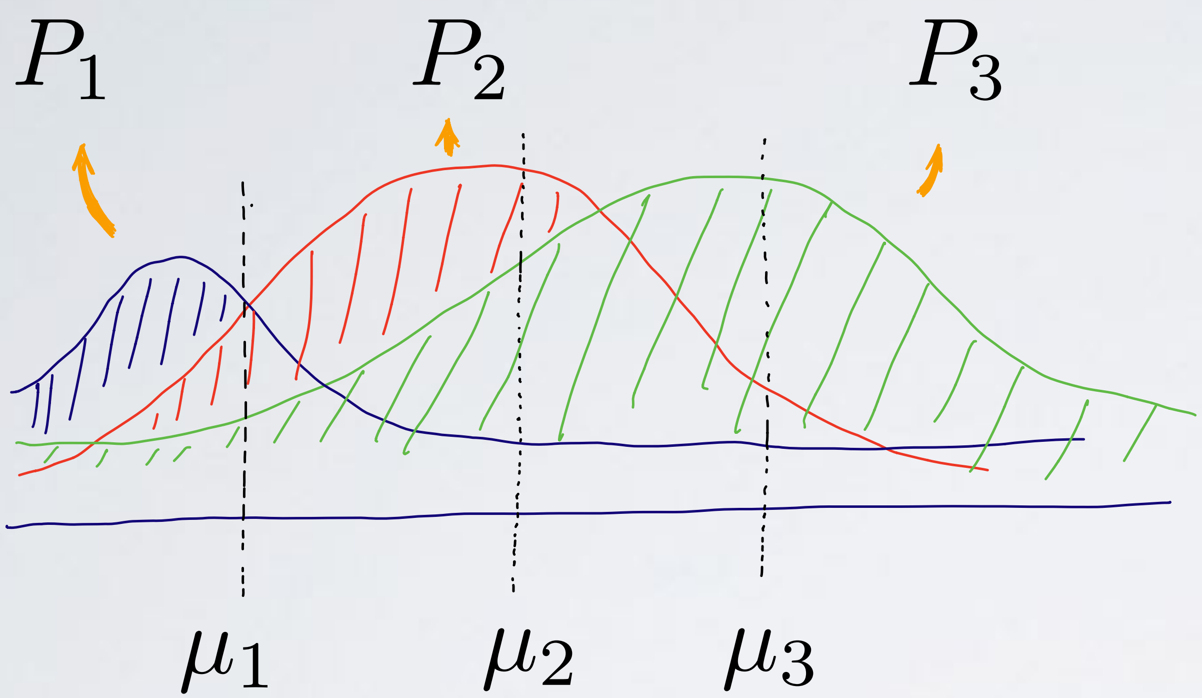

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$$X_{a_t} \sim P_{a_t}$$

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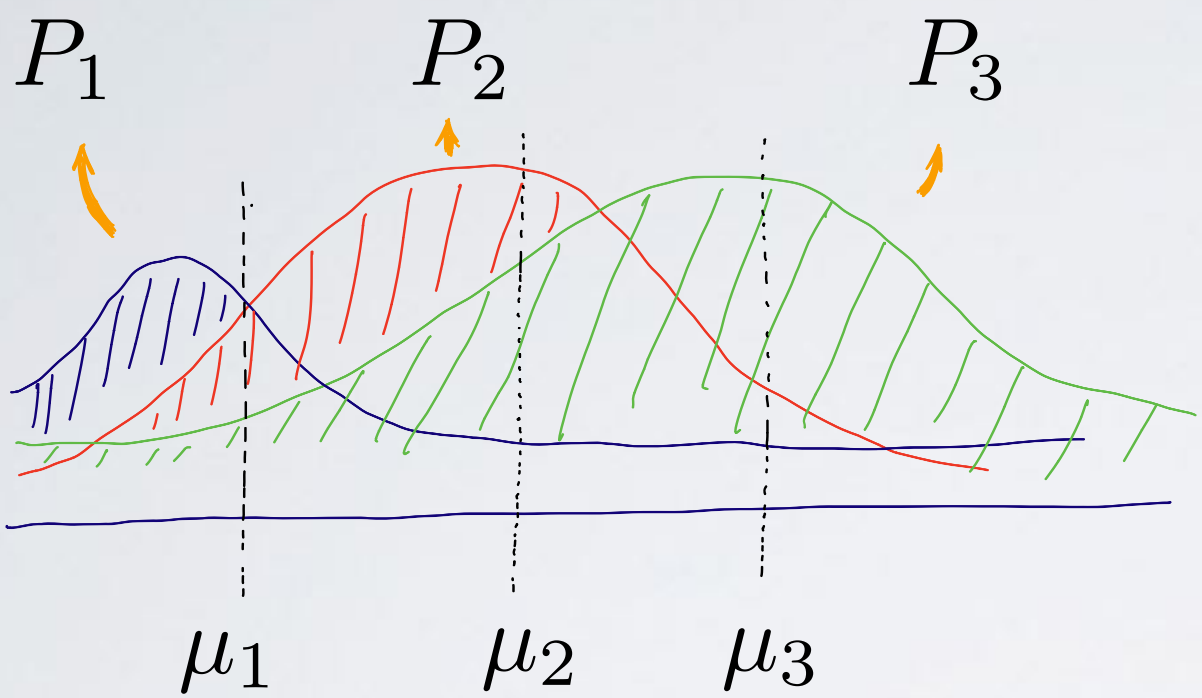

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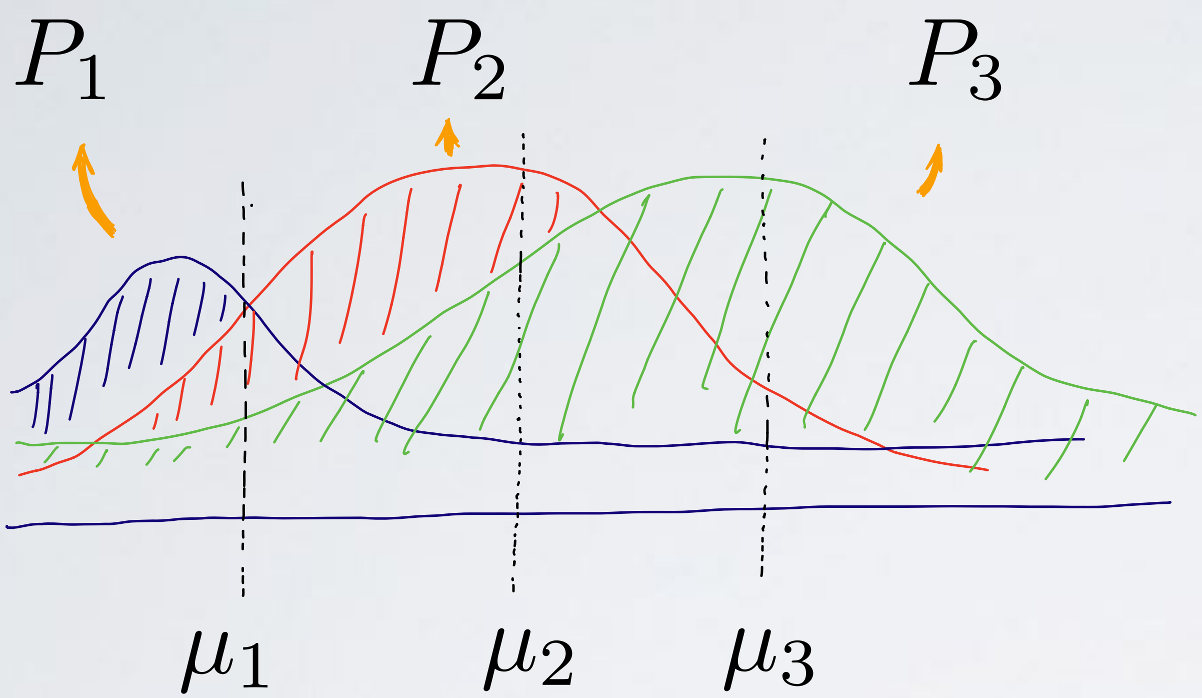

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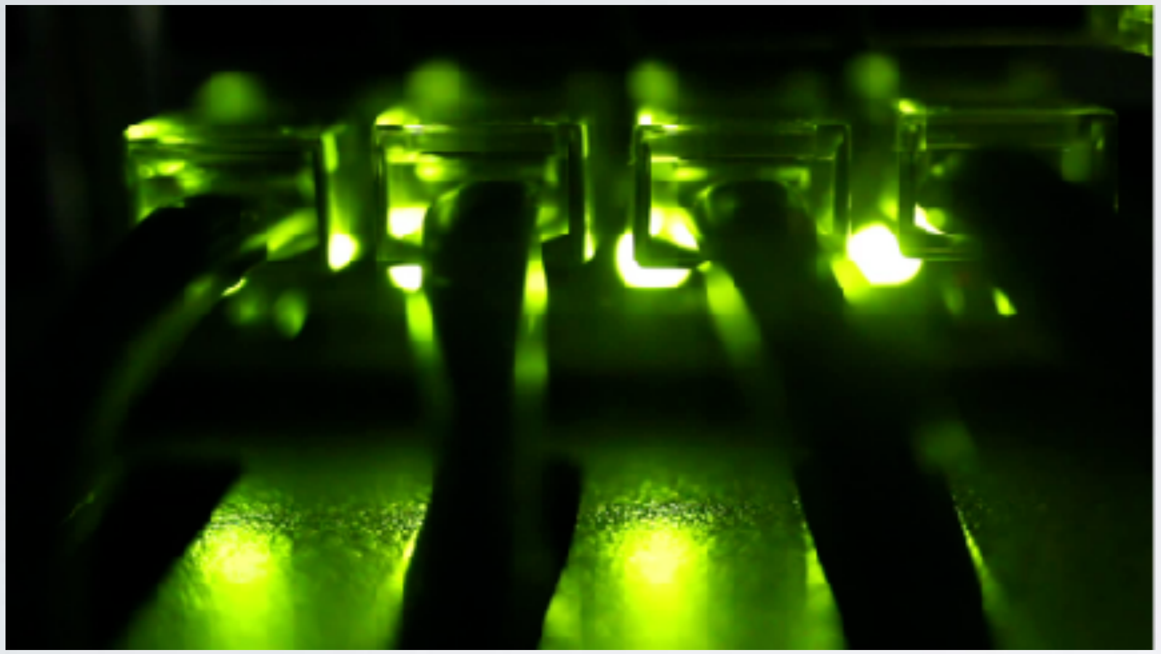
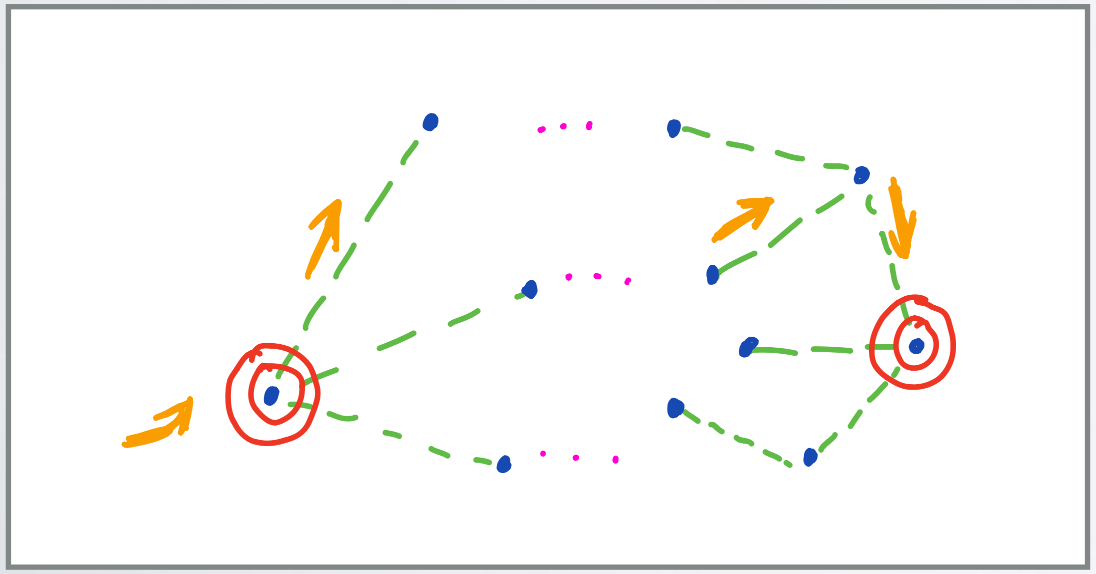
MAB regret $R(n) = \mathcal{O}(\sqrt{Kn\log(n)})$

[Auer et al. 2002]

Structured losses



Packet routing



Network (V, E)

Arms = Paths $a_t \in \mathcal{A} \subset \{0,1\}^E$

Loss = delay $w_t \in \mathcal{W} = [0,1]^E$

MAB regret

Exponential

$$R(n) = \mathcal{O}(\sqrt{|\text{num paths}| \cdot n \log(n)})$$

Delay is linear $\langle a_t, w_t \rangle$

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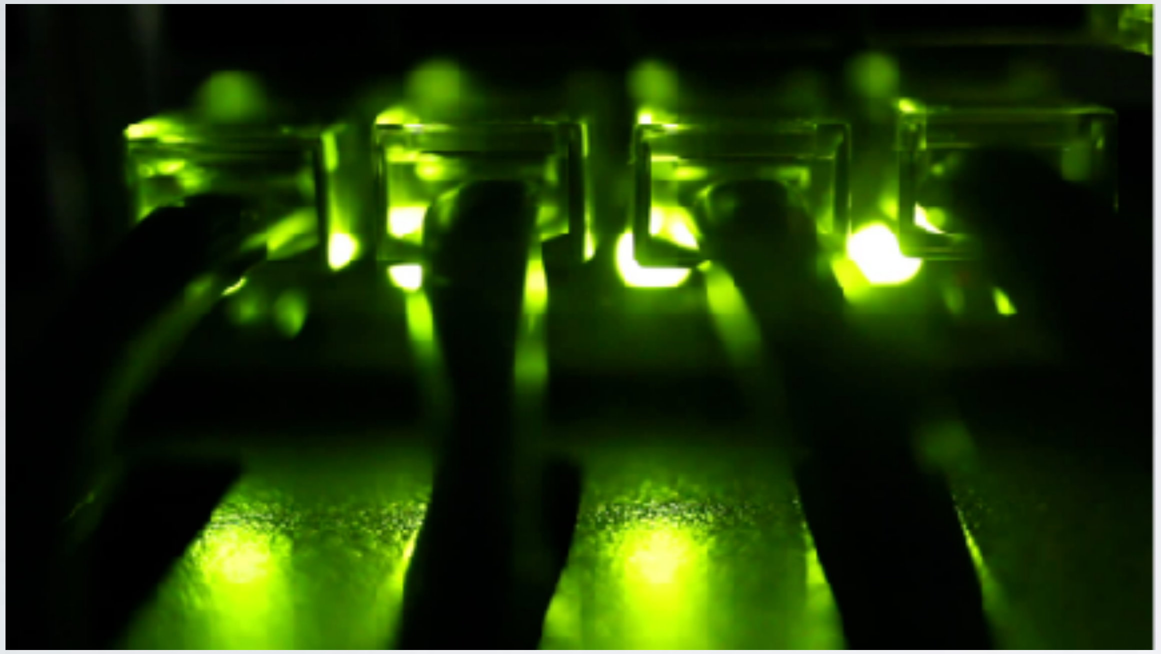

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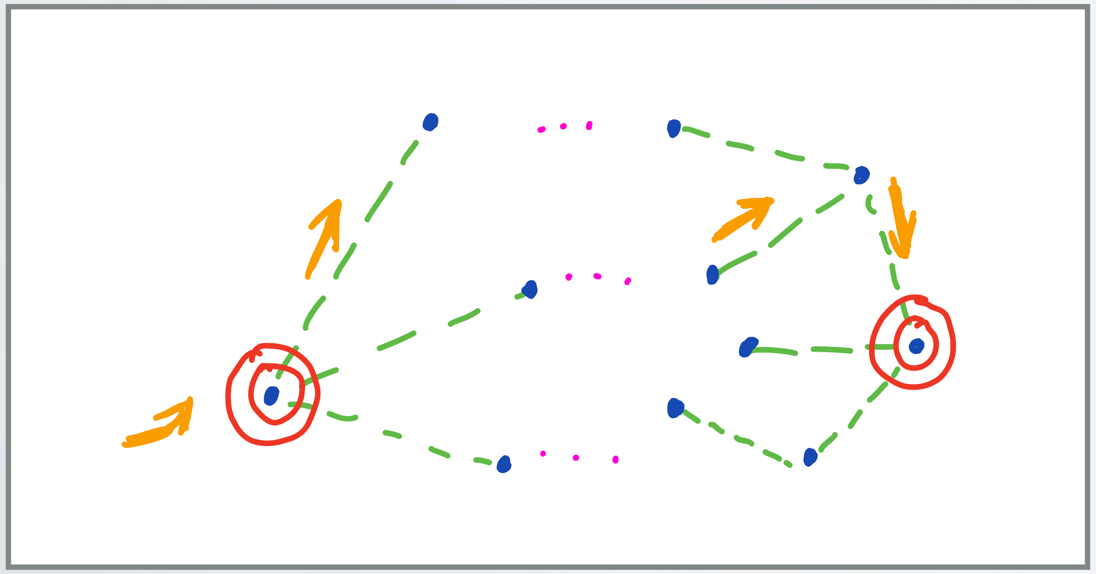

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Expected regret:

$$R(n) = \mathbb{E}\left[\sum_{t=1}^{n} \langle w_t, a_t \rangle - \inf_{a \in \mathcal{A}} \sum_{t=1}^{n} \langle w_t, a \rangle\right]$$

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MAB reduces to Linear Bandits

$$\mathcal{A} = \{e_1, \cdots, e_d\}, \qquad \mathcal{W} = [0, 1]^d$$

Exponential weights for adversarial linear bandits

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For $t = 1, \cdots, n$:

Sample mixture $a_t \sim p_t = \underbrace{(1-\gamma) q_t}_{\text{Exploitation}} + \underbrace{\gamma \nu}_{\text{Exploration}}$

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Update $\underbrace{q_t(a) \propto \exp(-\eta\langle \hat{w}_t, a\rangle) q_{t-1}(a)}_{\text{Exponential weights}}$

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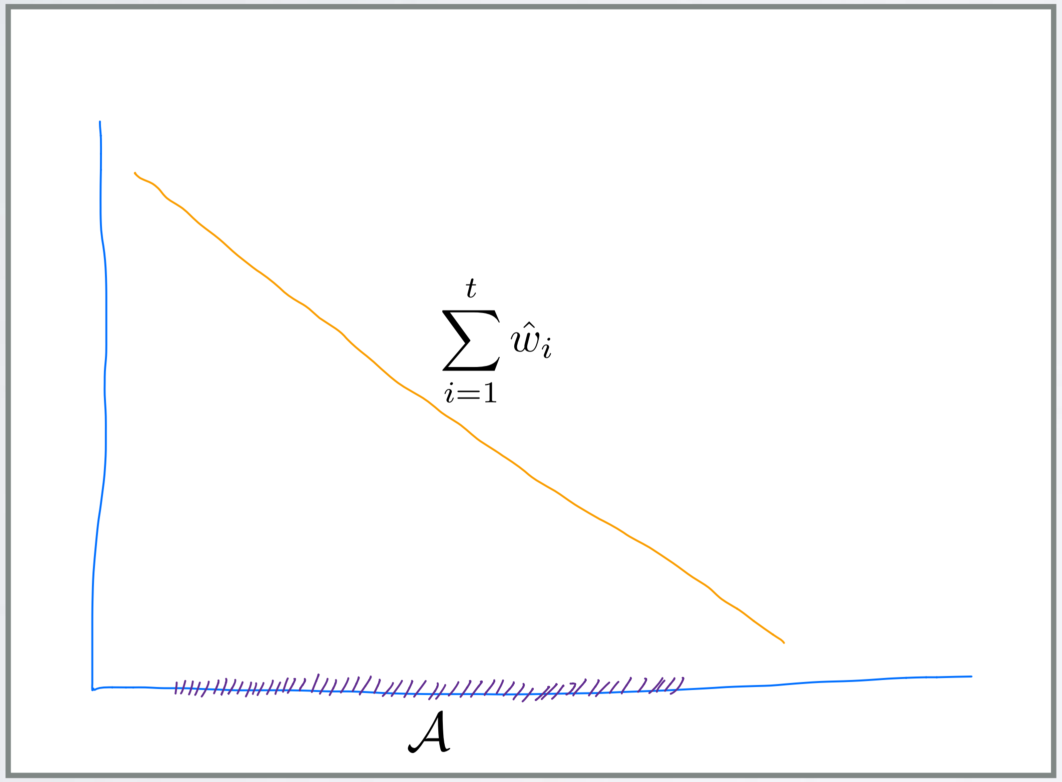

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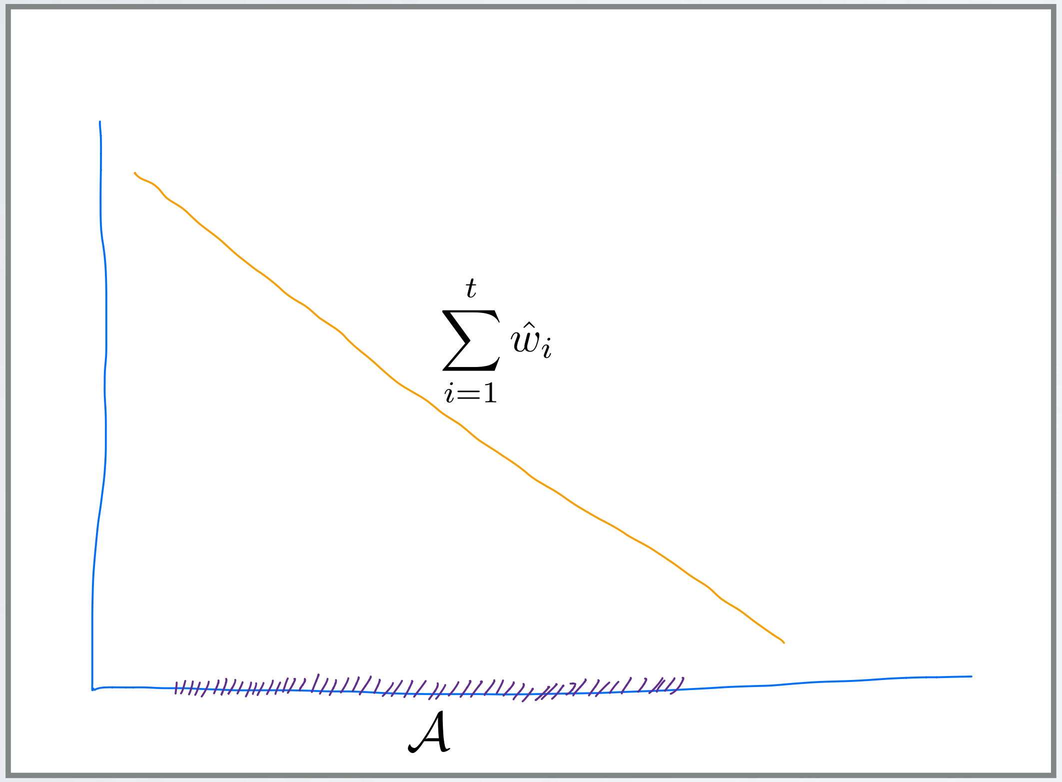

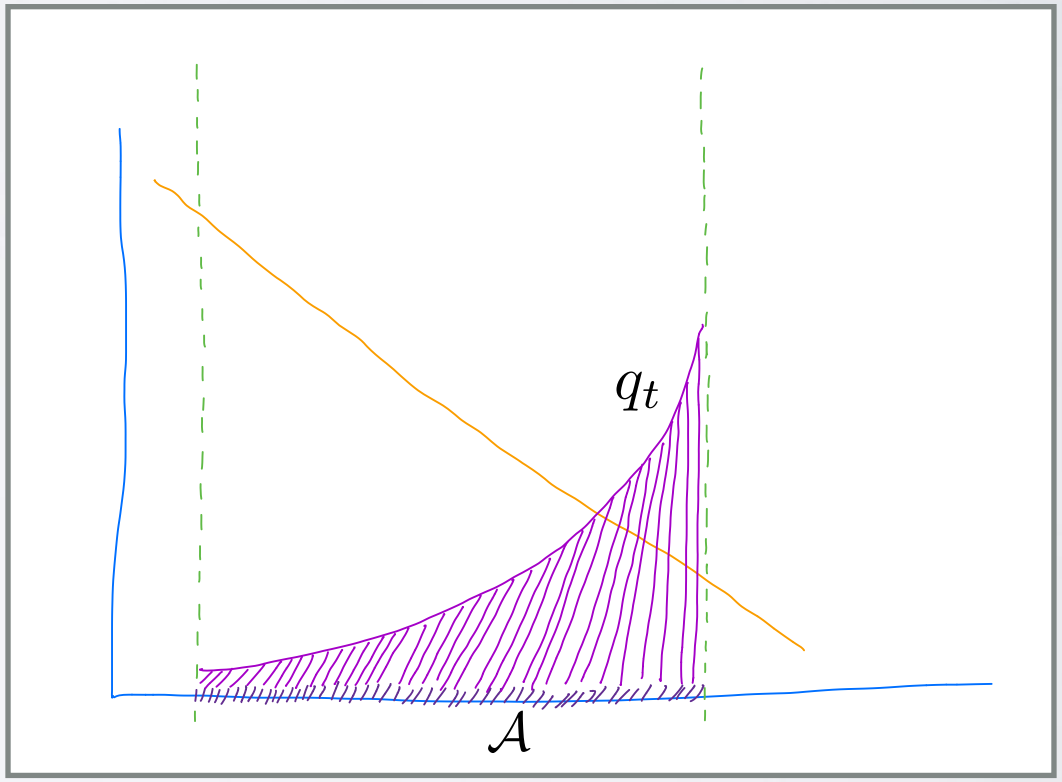

Unbiased estimator of the loss

Let $\Sigma_t = \mathbb{E}_{a \sim p_t}\left[aa^\top\right]$ and set $\hat{w}_t = (\Sigma_t)^{-1} a_t \langle w_t, a_t \rangle$

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$$\begin{aligned}
\mathbb{E}_{a_t \sim p_t}\left[\hat{w}_t | \mathcal{F}_{t-1}\right] &= \left(\mathbb{E}_{a \sim p_t}\left[aa^T\right]\right)^{-1} \mathbb{E}_{a_t \sim p_t}\left[a_t \langle w_t, a_t \rangle | \mathcal{F}_{t-1}\right] \\
&= \left(\mathbb{E}_{a \sim p_t}\left[aa^T\right]\right)^{-1} \mathbb{E}_{a_t \sim p_t}\left[a_t a_t^\top | \mathcal{F}_{t-1}\right] w_t \\
&= w_t
\end{aligned}$$

Linear bandits regret

Theorem. *(Linear Bandits Regret).*

[See for example Bubeck '11]

$$R(n) \leq \gamma n + \frac{\log(|\mathcal{A}|)}{\eta} + \eta \sum_{t=1}^{n} \mathbb{E}\mathbb{E}_{a \sim p_t}(\langle \hat{w}_t, a \rangle)^2$$

Exploration over Barycentric Spanner, [Dani, Hayes, Kakade '08]

$$\mathcal{O}(d\sqrt{n \log(|\mathcal{A}|)}) = \mathcal{O}(d^{3/2}\sqrt{n})$$

Uniform over $\mathcal{A}$, [Cesa-Bianchi, Lugosi, '12]

$$\mathcal{O}(\sqrt{dn \log(|\mathcal{A}|)}) = \mathcal{O}(d\sqrt{n})$$

John's distribution [Bubeck, Cesa-Bianchi, Kakade '12]

$$\mathcal{O}(d\sqrt{n})$$

Linear bandits regret
Dimension dependence

Variance bound:

$$\mathbb{E}\left[\mathbb{E}_{a_t \sim p_t}\left[(\langle \hat{w}_t, a\rangle)^2\right]\right] \leq d$$

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$$R(n) \leq \gamma n + \frac{\log(|\mathcal{A}|)}{\eta} + \underbrace{\eta \sum_{t=1}^{n} \mathbb{E}\mathbb{E}_{a \sim p_t}(\langle \hat{w}_t, a\rangle)^2}_{\leq \eta d n}$$

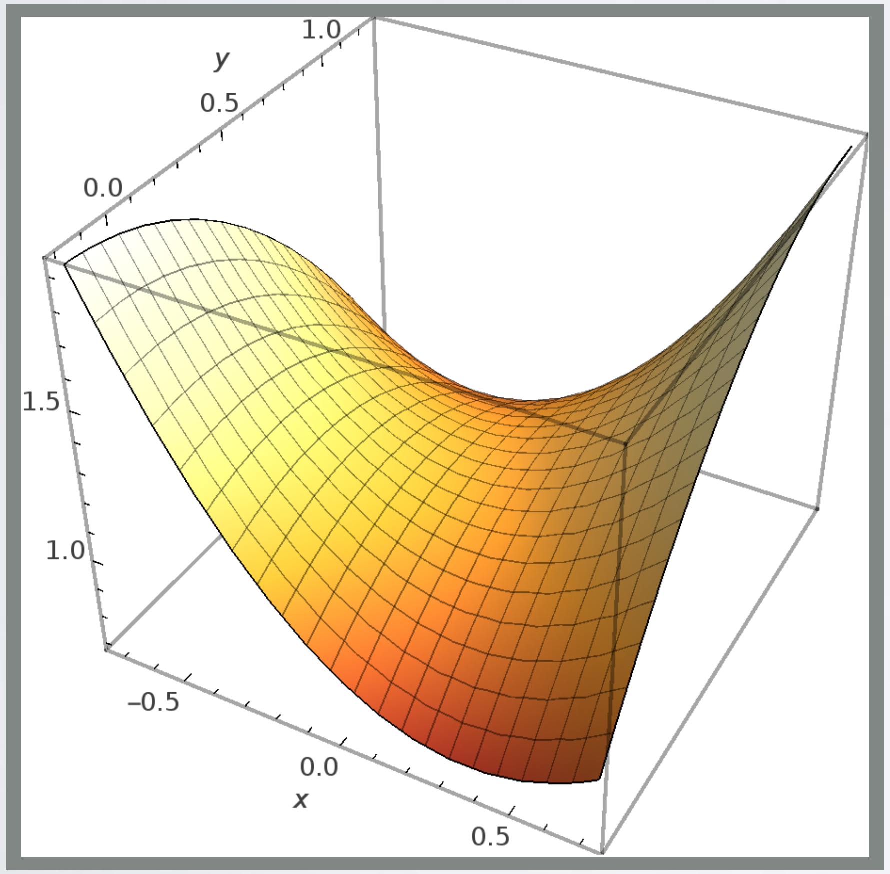
Recap

- Intro to Online Learning
- Linear Bandits
- **Kernel Bandits**

Online Quadratic losses

$$a_t \in \mathcal{A} = \{a \ \text{ s.t. } \ \|a\|_2 \leq 1\}$$

$$\ell_t(a) = \langle b_t, a \rangle + a^\top B_t a$$



$$z = x^2 - .5 * y^2 + x * y - .5 * x + .5y + 1$$

B_t Symmetric and possibly non convex

$$\min_{a \in \mathcal{A}} \ell_t(a)$$

Offline problem has polytime solution

Strong Duality

Peter Bartlett

Niladri Chatterji

Linearization of Quadratic losses

Quadratic losses are linear in the space of $\begin{pmatrix}\text{matrices}\\ \text{vector}\end{pmatrix}$

$$\ell(a) = \langle b_t, a\rangle + a^\top B_t a \quad \Rightarrow \quad \ell(a) = \left\langle \begin{pmatrix} B_t \\ b_t \end{pmatrix}, \begin{pmatrix} aa^\top \\ a \end{pmatrix} \right\rangle$$

We can use the linear bandits machinery

Exponential weights for quadratic bandits

Exponential weights for adversarial quadratic bandits

For $t = 1, \cdots, n$:

Sample mixture $a_t \sim p_t = \underbrace{(1-\gamma)q_t}_{\text{Exploitation}} + \underbrace{\gamma\nu}_{\text{Exploration}}$

See $\langle b_t, a_t \rangle + a_t^\top B_t a_t$

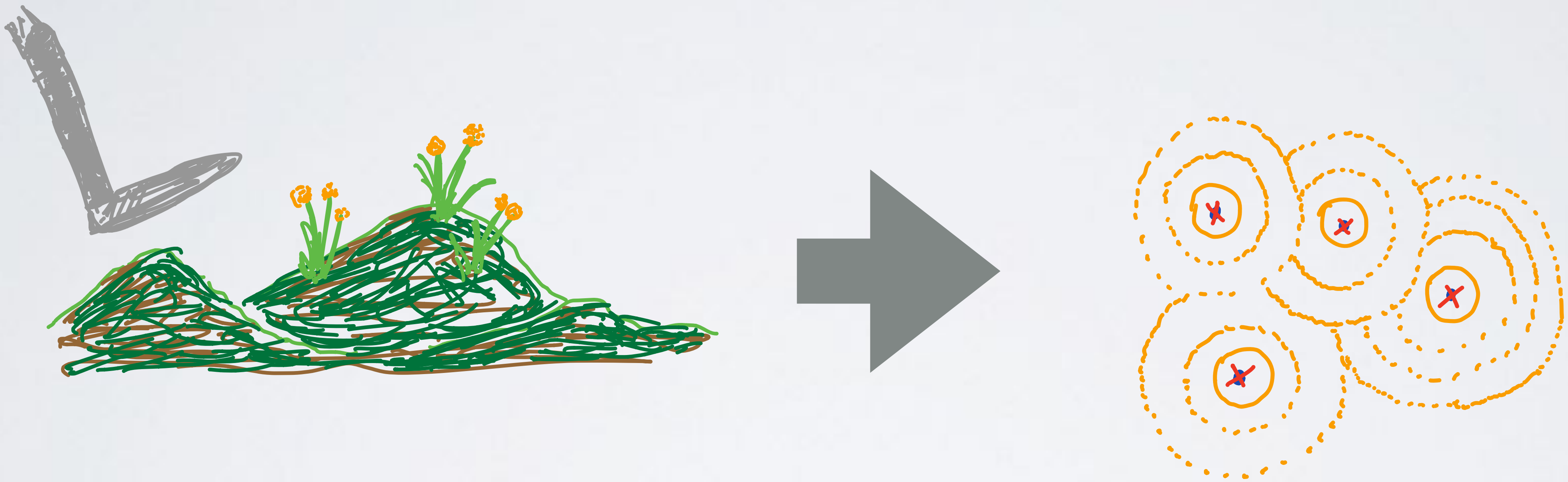
Build loss estimator $\begin{pmatrix} \hat{B}_t \\ \hat{b}_t \end{pmatrix}$

Update $q_t(a) \propto \underbrace{\exp(-\eta(\langle \hat{b}_t, a \rangle + a^\top \hat{B}_t a))q_{t-1}(a)}_{\text{Exponential weights}}$

Sampling is poly time

Beyond "Finite Dimensional" Losses

Evasion games:



Obstacle avoidance

$$\ell_t(a) = \exp(-\|a - w_t\|^2)$$

Gaussian kernel - Infinite dimensional

Space of Quadratics as a Reproducing Kernel Hilbert Space

Feature map
$x \to \Phi(x) \in \mathbb{R}^D$

Dot product
$\mathcal{K}(x, y) = \langle \Phi(x), \Phi(y) \rangle$

The Reproducing Kernel Hilbert Space $\mathcal{H}_\mathcal{K}$ of $\mathcal{K}$

Quadratics losses lie in an RKHS

$$\mathcal{K}(x, y) = \langle x, y \rangle + (\langle x, y \rangle)^2$$

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Main challenge:

Dimension of $\mathcal{H}_{\mathcal{K}}$ might be infinite.

Naive Linear regret $\mathcal{O}(\sqrt{dn \log(|\mathcal{A}|)})$

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Encouraging facts:

Kernel spaces are "small"

Towards an Algorithm

Algorithm Strategy:

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1) Construct an $m < \infty$ dimensional proxy kernel that *uniformly* approximates the original kernel over $\mathcal{A} \times \mathcal{A}$.

$$\mathcal{K}_m(x, y) \approx \mathcal{K}(x, y)$$

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Kernel functions are "small"

Kernel function evaluations can be uniformly approximated by a small number of basis functions.

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Mercer's Theorem

There exist functions $\{\phi_i\}_{i=1}^{\infty}$ and nonnegative values $\{\mu_i\}_{i=1}^{\infty}$ such that:

$$\mathcal{K}(x, y) = \sum_{i=1}^{\infty} \mu_i \phi_i(x) \phi_i(y) \qquad \forall x, y \in \mathcal{A} \times \mathcal{A}$$

Eigendecay

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Gaussian Kernel

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Sobolev Kernel

$\mathcal{K}(x, y) = \min(x, y)$

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Exponential decay

$\mu_j \leq Ce^{-\beta j}$

$\{\phi_j(x)\} = \{\sin(j\pi x), \cos(j\pi x)\}$

$\mu_j \approx e^{-cj\log(j)}$

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Polynomial decay
$\mu_j \leq C j^{-\beta}$

$\phi_j(x) \approx \sin\left(\frac{2j\pi x}{2}\right)$

$\mu_j \approx \frac{1}{j^2}$

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$$\mathcal{K}(x,y) = \sum_{i=1}^{\infty} \mu_i \phi_i(x)\phi_i(y)$$

<u>*Gaussian Kernel*</u>

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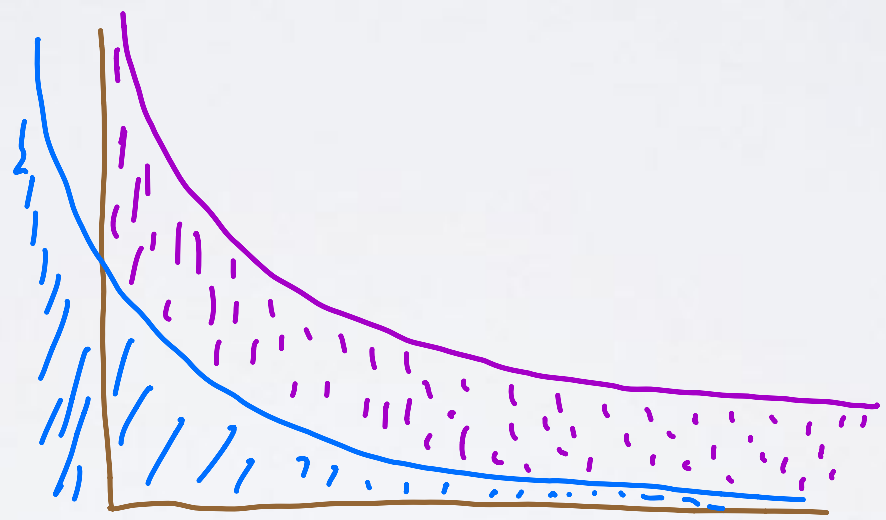

<u>*Sobolev Kernel*</u>

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Construction of a finite dimensional Proxy Kernel

Eigenfunctions $\{\phi_i\}_{i=1}^{\infty}$ with eigenvalues $\{\mu_i\}_{i=1}^{\infty}$

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$$\mathcal{K}^o(x,y) = \sum_{i=1}^{m} \mu_i \phi_i(x) \phi_i(y)$$

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Deterministic Proxy Kernel

$$\mathcal{K}^o(x,y) = \sum_{i=1}^{m} \mu_i \phi_i(x) \phi_i(y)$$

Building proxy Kernel with samples
Kernel PCA

Exponential weights for adversarial kernel bandits

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Sampling might not be poly time

Biased loss estimates

Let $\Sigma_m^{(t)} = \mathbb{E}_{a\sim p_t}\left[\Phi_m(a)\Phi_m(a)^\top\right]$ and set $\hat{w}_t := \mathcal{K}(a_t, y_t)\left((\Sigma_m^{(t)})^{-1}\Phi_m(a_t)\right)$

$\hat{w}_t$ is biased:

$$\begin{aligned}\mathbb{E}_{a_t\sim p_t}\left[\hat{w}_t|\mathcal{F}_{t-1}\right] &= \mathbb{E}\left[\mathcal{K}(a_t, y_t)\left((\Sigma_m^{(t)})^{-1}\Phi_m(a_t)\right)\Big|\mathcal{F}_{t-1}\right] \\ &= \Phi_m(y_t) + \mathbb{E}\left[\underbrace{\left(\mathcal{K}(a_t, y_t) - \hat{\mathcal{K}}_m(a_t, y_t)\right)\left((\Sigma_m^{(t)})^{-1}\Phi_m(a_t)\right)}_{=:\xi_t,\text{ the bias}}\Big|\mathcal{F}_{t-1}\right]\end{aligned}$$

Kernel Bandits Regret

Expected regret

$$R(n) = \mathbb{E}\left[\sum_{t=1}^{n} \mathcal{K}(w_t, a_t) - \inf_{a \in \mathcal{A}} \sum_{t=1}^{n} \mathcal{K}(w_t, a^*)\right]$$

Theorem.

$$R(n) \leq 2\gamma n + \underbrace{\eta m n + \frac{2\epsilon n}{\eta}}_{Bias\ \ variance} + 2\epsilon n + \frac{1}{\eta}\log(|\mathcal{A}|).$$

ξ_t bias

$\mathcal{K}_m$ game vs $\mathcal{K}$ game

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Corollary.

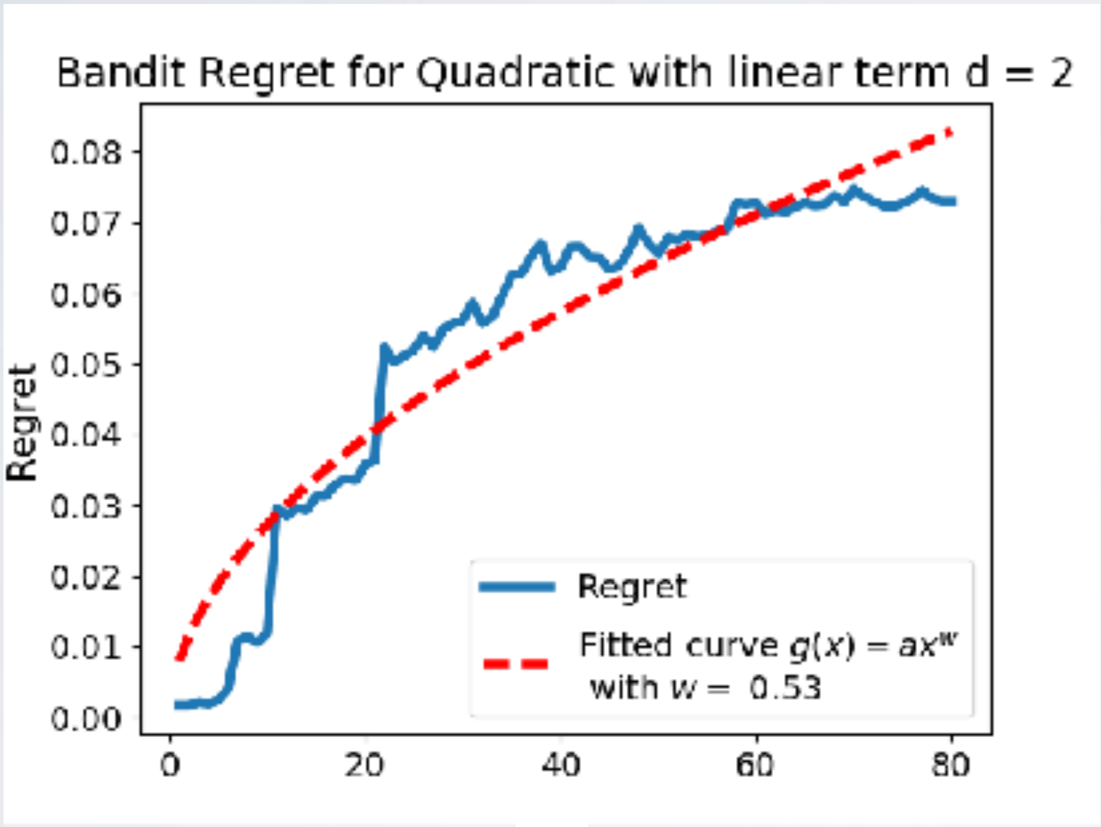
1 Polynomial Decay. $\mu_j \leq C j^{-\beta}$ *and* $\beta > 2$*:*

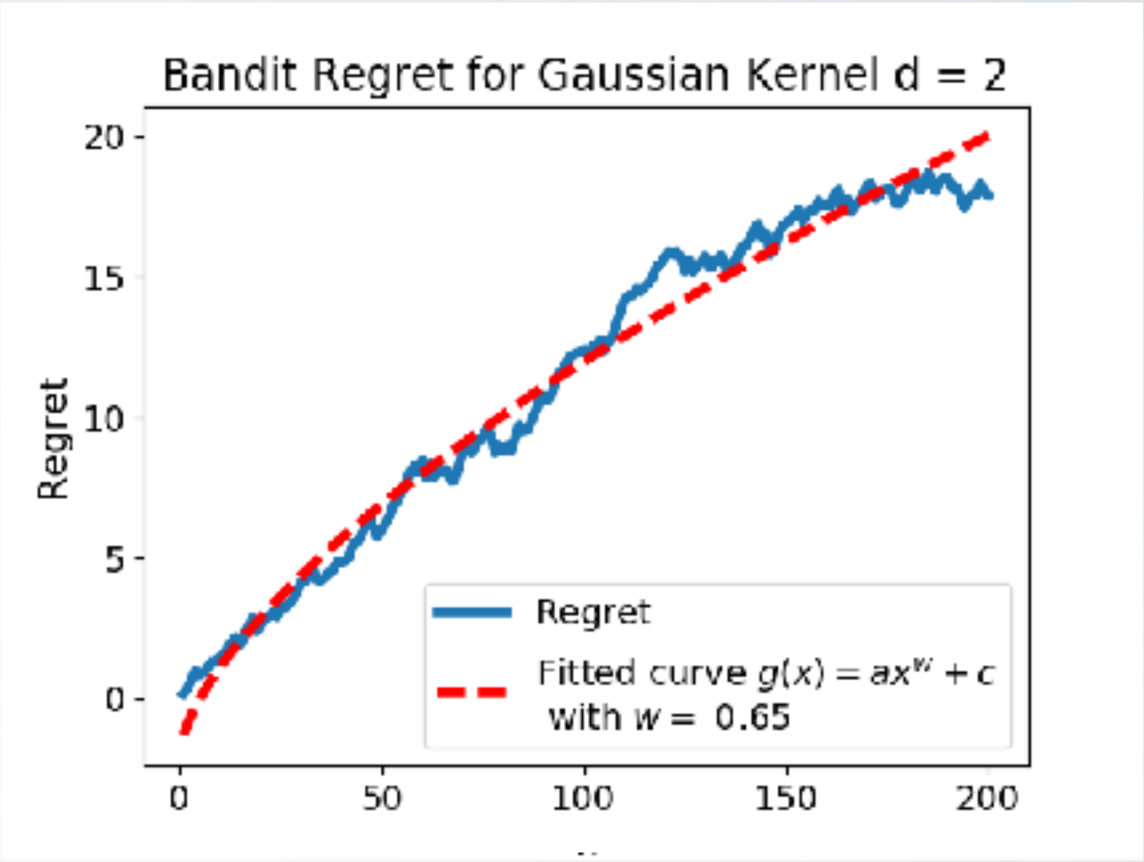
$$R(n) \leq \mathcal{O}\left(\log(|\mathcal{A}|)^{\frac{\beta-2}{2(\beta-1)}} n^{\frac{\beta}{2(\beta-1)}}\right)$$

2 Exponential Decay. $\mu_j \leq C e^{-\beta j}$*:*

$$R(n) \leq \mathcal{O}\left(\sqrt{\log(|\mathcal{A}|)\log(n)n}\right)$$

Experiments





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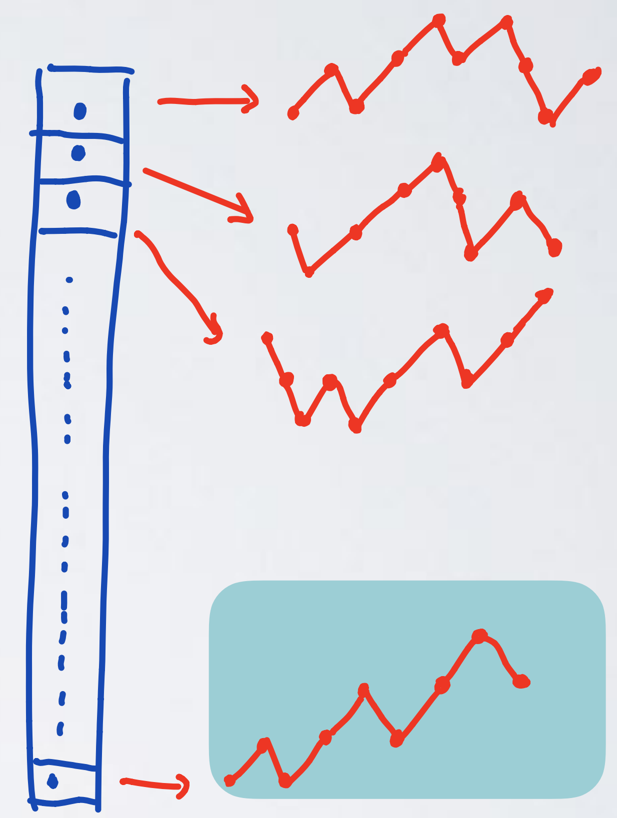

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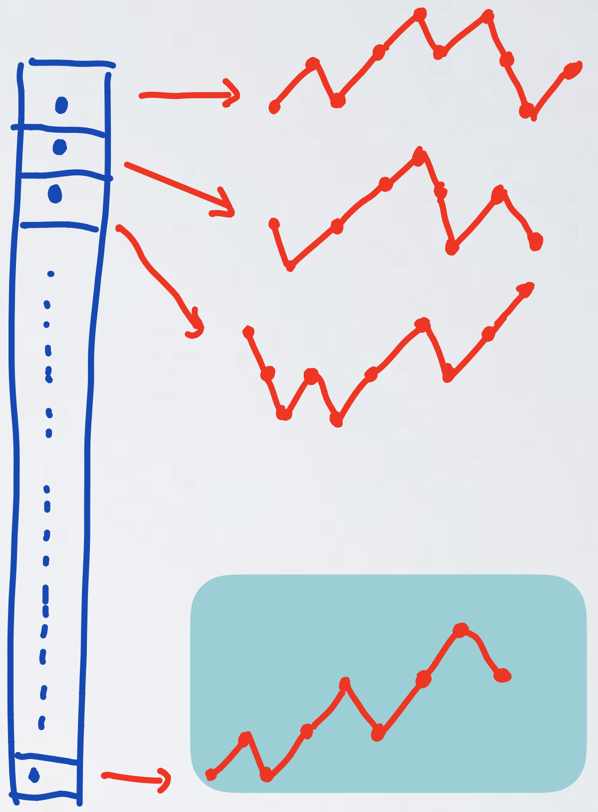

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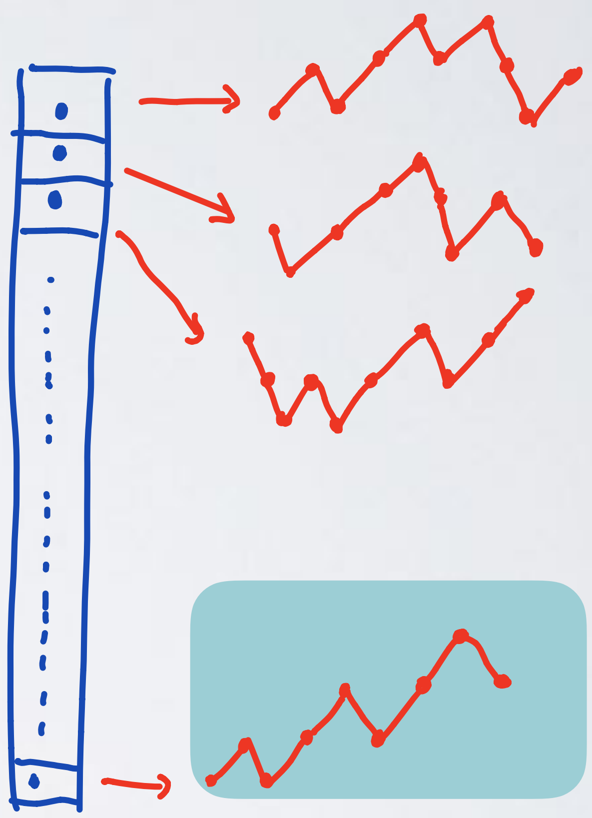

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Final thoughts

Thanks for your attention