PAC Learnability of Node Functions in Networked Dynamical Systems

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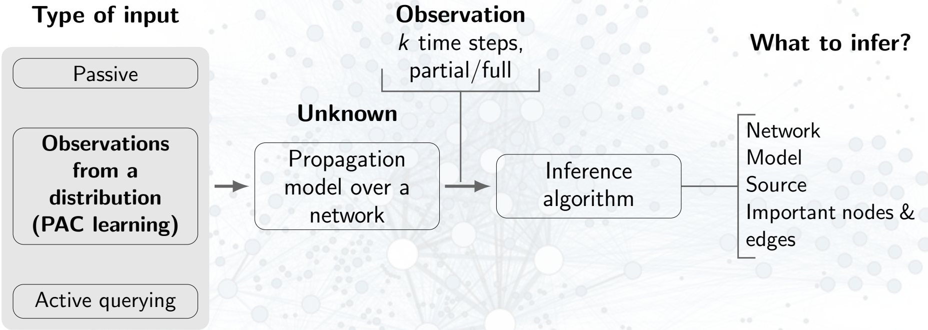
Inferring network propagation models

- Inferring network dynamical systems is a broad and well-studied area.



We consider the problem of inferring the node functions of a networked dynamical system.

- Observation model: Probably Approximately Correct (**PAC**) learning
- Model class: **Threshold dynamical systems**

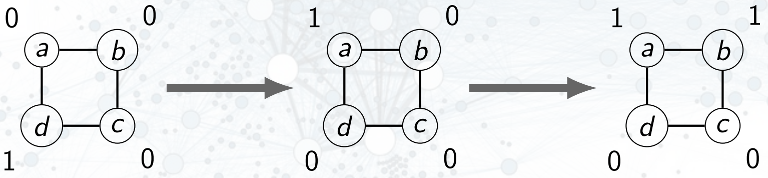
Motivation and previous work

- PAC learning network dynamical systems:
  - Learning influence functions of nodes in stochastic networked dynamical systems [Narasimhan et al., 2015; He et al., 2016].
  - Extensive research on PAC learning threshold functions, and in general, Boolean functions [Hellerstein & Servedio 2007].
- Practical Use of Threshold models:
  - Widespread application in modeling protests, information diffusion (e.g., word of mouth, social media), adoption of practices (e.g., contraception, innovation), transmission of emotions, etc. (Granovetter 1978).
  - Social science network experiments (Centola 2010).
  - General inference: (González-Bailón et al. 2011; Romero, Meeder, and Kleinberg 2011) present methods to infer thresholds from social media data.

Threshold propagation model

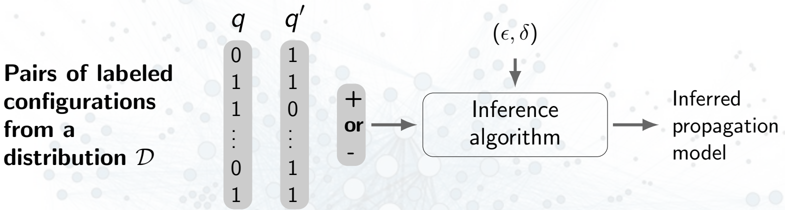
- Closed neighborhood of a vertex v: $N[v]$
- Every node is associated with a threshold: $t(v)$

$$q_{i+1}(v) = \begin{cases} 1, & \sum_{v' \in N[v]} q_i(v') \geq t(v) \\ 0, & \text{otherwise} \end{cases}$$



$$t(a) = 1,\ t(b) = 1,\ t(c) = 2,\ t(d) = 2$$

Probably Approximately Correct (PAC) learning framework



- $+$ means q' is the successor of q. Otherwise, it is not.
- User knows:
  - Network (undirected, unweighted)
  - Concept class: threshold functions

Questions of interest

- Are threshold dynamical systems efficiently learnable?
- Sample complexity: How many examples (i.e., pairs of configurations) are sufficient to infer the dynamical system?
- Is there an efficient learning algorithm?
- How do these algorithms perform on real-world networks?

Results

Sample complexity

Threshold dynamical systems are PAC learnable.

- Upper bound on sample complexity $\mathcal{M}(\epsilon, \delta)$:

  $$\mathcal{M}(\epsilon, \delta) \ \leq \ \tfrac{1}{\epsilon}\big(n \log(d_{\text{avg}} + 3) + \log(1/\delta)\big).$$

  We also extend the bound to other classes of threshold functions.

- Lower bounds on sample complexity:
  - $\Omega(n/\epsilon)$ using Vapnis-Chervonenkis (VC) dimension of the hypothesis space of threshold functions.
  - It is within a factor $O(\log n)$ of the upper bound.
  - Tight example: When the underlying graph is a **clique**, the VC dimension of the hypothesis space is $\leq n + 1$.

Results

Algorithmic efficiency

Hardness of learning depends on negative examples.

- When there are both positive and negative examples, the hypothesis class of threshold functions is not *efficiently* PAC learnable, unless the complexity classes **NP** and **RP** (Randomized Polynomial time) coincide.

Efficient learning algorithms:

- When there are only positive examples, we present an algorithm which learns in time $O(|\mathcal{E}|n)$, where $\mathcal{E}$ is the set of examples and n is the number of nodes.
- **Exact algorithm**: When a set $\mathcal{E}_N$ of negative examples is also given, we present a dynamic programming algorithm that learns in time $O(2^{|\mathcal{E}_N|}\mathrm{poly}(n))$, which is polynomial when $|\mathcal{E}_N| = O(\log n)$.
- **Approximation algorithm**: Using submodular function maximization under matroid constraints, we present an efficient learner which is consistent with all the positive examples and at least $(1 - 1/e)$ fraction of the negative examples.

Results

Experiments

Network	Properties			
	n	$\|E\|$	d_{ave}	d_{max}
Jazz	198	2742	27.70	100
NRV	769	4551	11.84	20
euEmall	986	16064	32.58	345
Ran Reg[a,1]	11–1000	$n\ d_{avg}/2$	10	10
Scl free[a,2]	20–1000	$\sim n\ d_{avg}/2$	9.5–9.9	13–149
Cliques[3]	400	$n_q n_c(n_c-1)/2$	$n_c - 1$	$n_c - 1$

Accuracy and sample complexity

- Effect of graph size
- Effect of graph density
- Effect of distributions for sampling configurations