Fast Algorithm for Generalized Multinomial Models with Ranking Data

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Generalized multinomial models

Consider d basic cells $c_1, \ldots, c_d$, where c_i is assigned with cell probability p_i ($\sum_{i=1}^d p_i = 1$). Suppose cell c_i is chosen for a_i times ($i = 1, \ldots, d$), then the log-likelihood function is

$$\ell(\mathbf{p}) = \sum_{i=1}^{d} a_i \log p_i. \tag{1}$$

Completeness condition: Sets of basic cells for selection are always $\{c_1\}, \ldots, \{c_d\}$. (choose 1 from d)

If completeness condition is violated:

- Union of sets for selection includes only a fraction of basic cells. (choose 1 from $k < d$)
- Sets for selection consists of more than one basic cell. (choose $l > 1$ from d)

$\rightarrow$ Incomplete multinomial models.

Log-likelihood function

$$\ell(\mathbf{p}) = \sum_{j=1}^{n} \Big\{ \log(\sum_{c_i \in A^j} p_i) - \log(\sum_{c_i \in \mathcal{C}^j} p_i) \Big\}, \qquad (2)$$

where $\mathcal{C}^j$ is the union of sets for selection, A^j is the selected set in j-th record.

Examples:

- Placett–Luce model [3, 4];
- Bradley–Terry model [1];
- Contingency table model [2].

Markov chain based algorithm

Denote $W_i = \{j : c_i \in A^j\}$ and $L_i = \{j : c_i \in (\mathcal{C}^j \backslash A^j)\}$,
$q_j^+ = \sum_{c_i \in A^j} p_i$ and $q_j^* = \sum_{c_i \in \mathcal{C}^j} p_i$,

$$\ell(\mathbf{p}) = \sum_{j=1}^{n} \Big\{ \log(q_j^+) - \log(q_j^*) \Big\}.$$

Letting $\frac{\partial \ell(\mathbf{p})}{\partial p_i} = 0$, we have

$$\sum_{i' \neq i} p_{i'} \left[\sum_{j \in W_i \cap L_{i'}} \frac{p_i}{q_j^+ q_j^*} \right] = \sum_{i' \neq i} p_i \left[\sum_{j \in L_i \cap W_{i'}} \frac{p_{i'}}{q_j^+ q_j^*} \right], \qquad (i = 1, \ldots, d). \tag{3}$$

Markov chain based algorithm

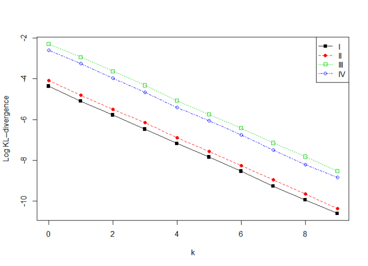
Algorithm 1 Markov chain based algorithm

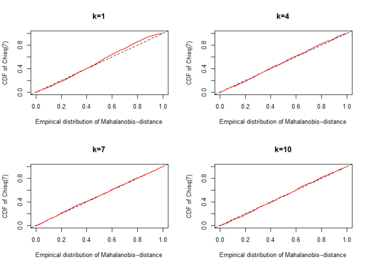
Input: Observations $\left\{(A^j, \mathcal{C}^j) : j = 1, \ldots, n\right\}$ and calculate $\{W_i, L_i\}$ for each c_i.
Initialize $\mathbf{p} = (1/d, \ldots, 1/d)^T$.
Initialize $\mathbf{\Sigma}(\mathbf{p}) = \mathbf{0}_{d \times d}$.
repeat
  for $i \in \{1, \ldots, d\}$ **do**
    for $i' \in \{1, \ldots, d\} \backslash \{i\}$ **do**
      Compute

$$\sigma_{ii'}(\mathbf{p}) \leftarrow \sum_{j \in L_i \cap W_{i'}} \frac{p_{i'}}{q_j^+ q_j^*}$$

    end for
  end for
  Compute $\sigma_{ii}(\mathbf{p})$ $(i = 1, \ldots, d)$ and then normalize $\mathbf{\Sigma}(\mathbf{p})$ so that $\forall i, \sum_{i'=1}^{d} \sigma_{ii'}(\mathbf{p}) = 1$.
  $\mathbf{p} \leftarrow T(\mathbf{p})$ under the transition matrix $\mathbf{\Sigma}(\mathbf{p})$.
until convergence.

Experiments: Convergence





Estimator obtained by our algorithm is close to the MLE, indicating our algorithm's convergence to the MLE.

Experiments: Convergence rate

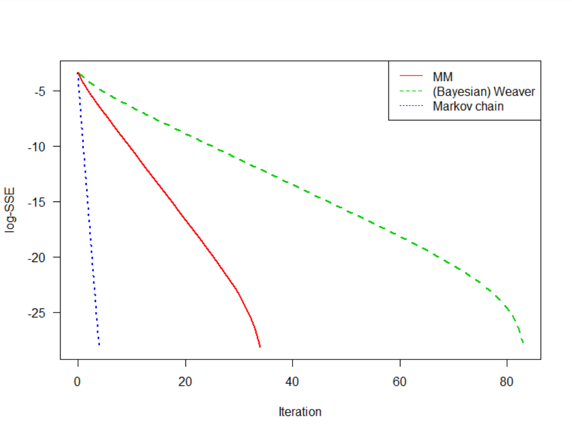
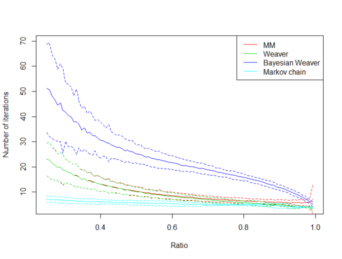


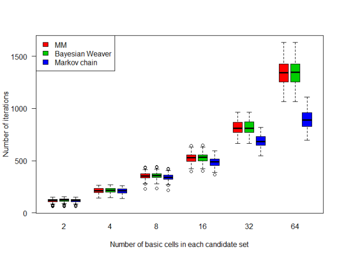
Figure 1: Path of iterations for three algorithms on sushi data

Experiments: Computational efficiency

- Choose 1 from $k < d$:



- Choose $l > 1$ from d:



Discussions

- Our algorithm obtain the MLE efficiently than existing methods.
- Further improvement. (Especially in situation "choose $l > 1$ from d")

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Thank you for listening.