# SATNet:

## Bridging deep learning and logical reasoning using a differentiable satisfiability solver

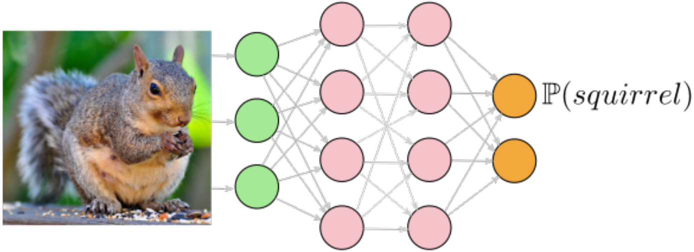
Po-Wei Wang [1] Priya L. Donti [1] Bryan Wilder [2] J. Zico Kolter [1,3]

[1] School of Computer Science, Carnegie Mellon University

[2] School of Engineering and Applied Sciences, Harvard University

[3] Bosch Center for Artificial Intelligence

# Integrating deep learning and logic




**Deep Learning**

No constraints on output

Differentiable

Solved via gradient optimizers

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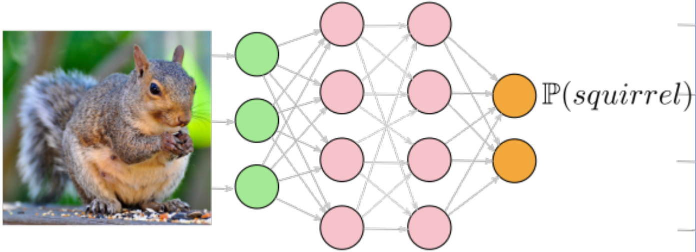


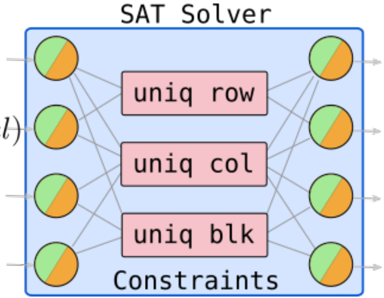


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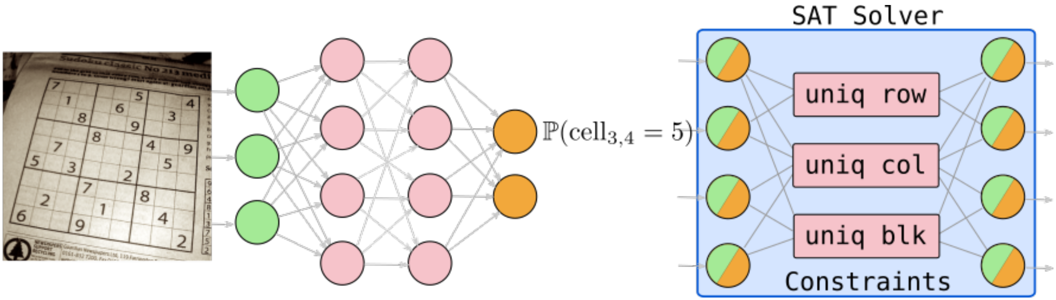


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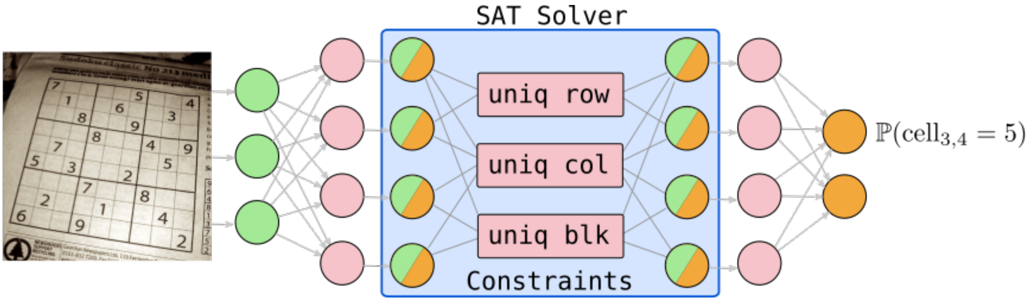


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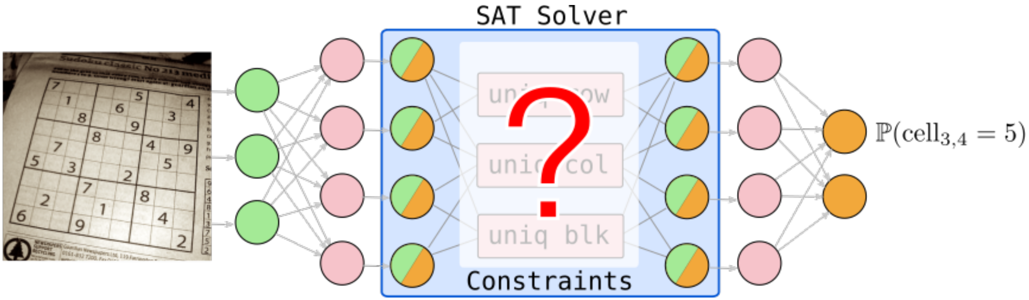


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Not about using DL and SAT in a multi-staged manner

- doing so requires prior knowledge on the stucture and constraints
- further, current SAT solvers cannot accept **probability inputs**

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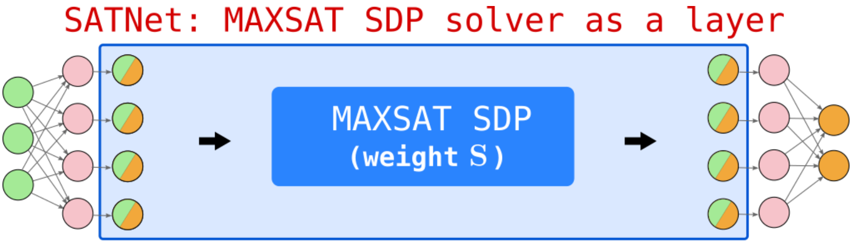
- A layer that enables end-to-end learning of **both** the **constraints** and **solutions** of logic problems within deep networks...
- A smoothed **differentiable (maximum) satisfiability solver** that can be integrated into the loop of deep learning systems.



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Example SAT problem:

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Typical SAT: Clause matrix given, find satisfying assignment

Our setting: Clause matrix is parameters of the layer (to be learned)

# MAXSAT Problem

MAXSAT is the optimization variant of SAT solving

**SAT:** Find feasible $v_i$ s.t. $v_2 \wedge (v_1 \vee \neg v_2) \wedge (v_2 \vee \neg v_3)$

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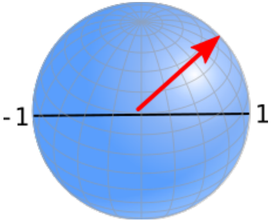


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Relax the binary variables to smooth & continuous spheres

$$v_i \in \{+1, -1\} \xrightarrow{equiv} |v_i| = 1,\ v_i \in \mathbb{R}^1 \xrightarrow{relax} \|v_i\| = 1,\ v_i \in \mathbb{R}^k$$

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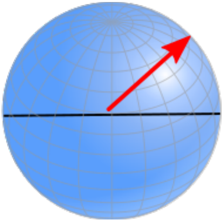


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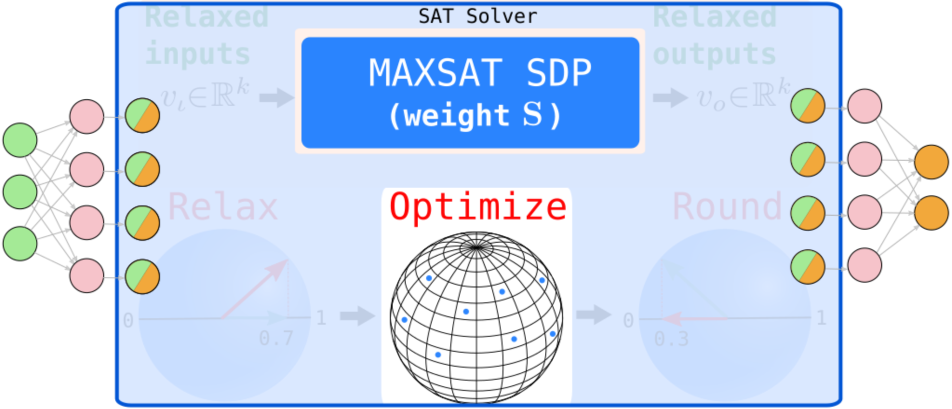
**Semidefinite relaxation** (Goemans-Williamson, 1995), $X = V^T V$

$$\text{minimize}\ \langle S^T S, X \rangle,\ \text{s.t. } X \succeq 0,\ \text{diag}(X) = 1.$$

# SATNet: MAXSAT SDP as a layer



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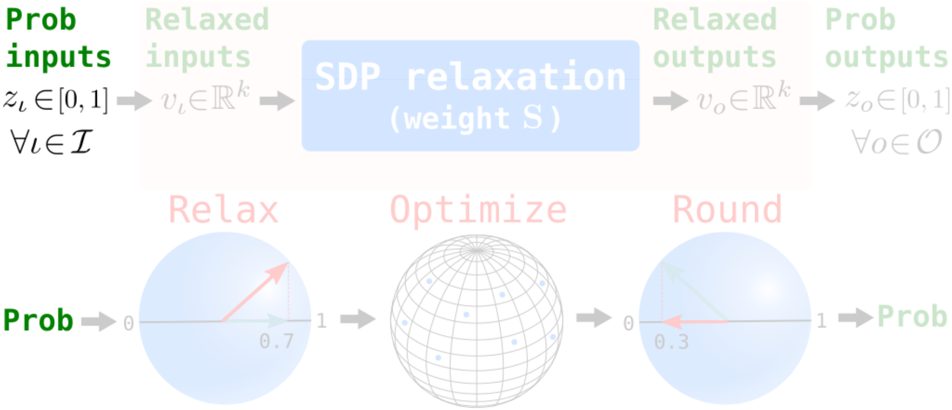

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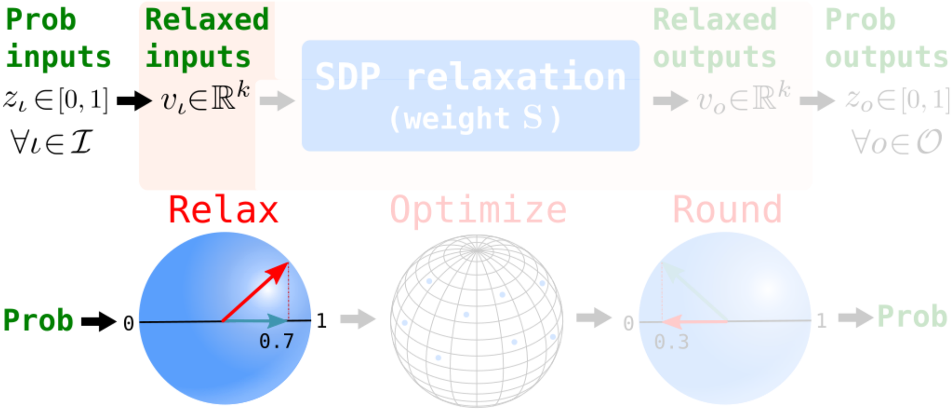

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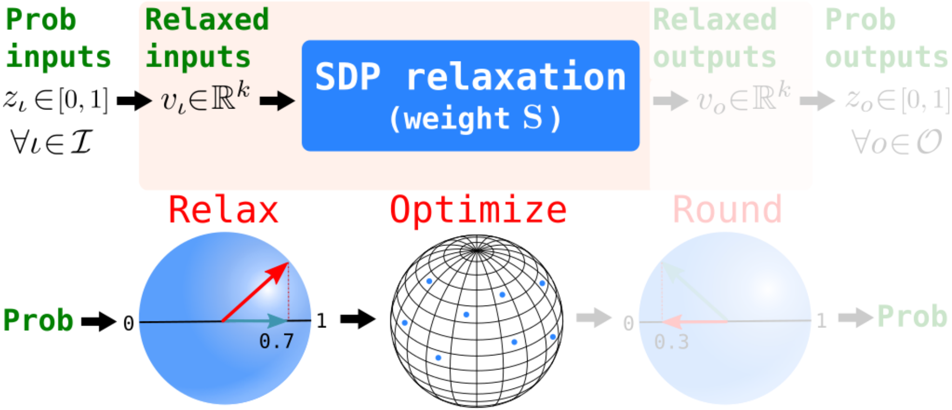

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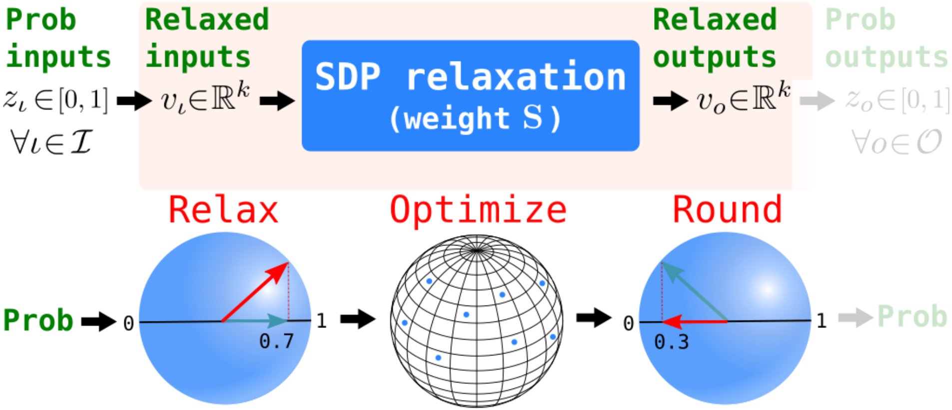

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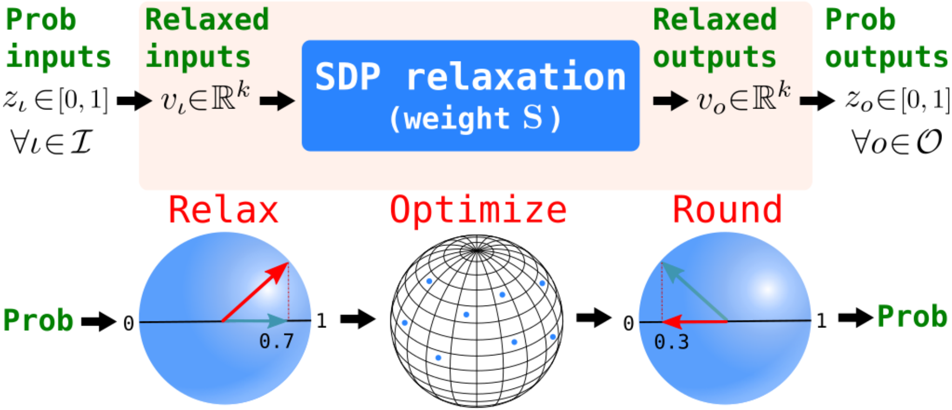

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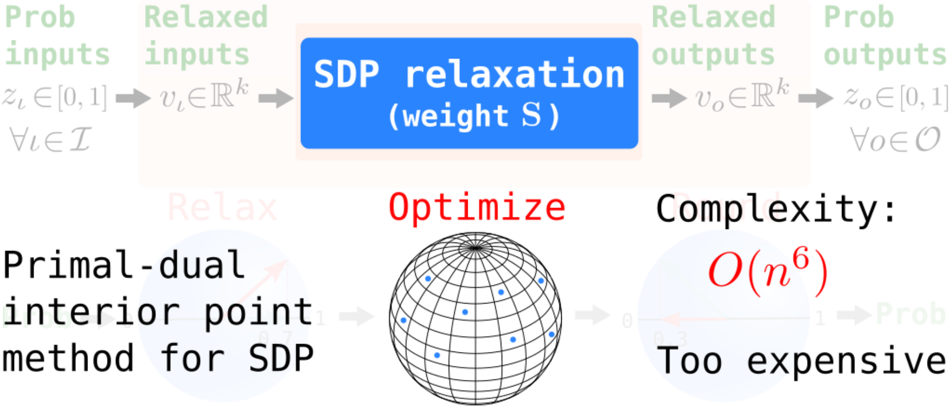

# Fast solution to MAXSAT SDP approximation

Efficiently solve via low-rank factorization $X = V^T V$, $V \in \mathbb{R}^{k \times n}$, $\|v_i\| = 1$ (a.k.a. Burer-Monteiro method), and block coordinate descent iters

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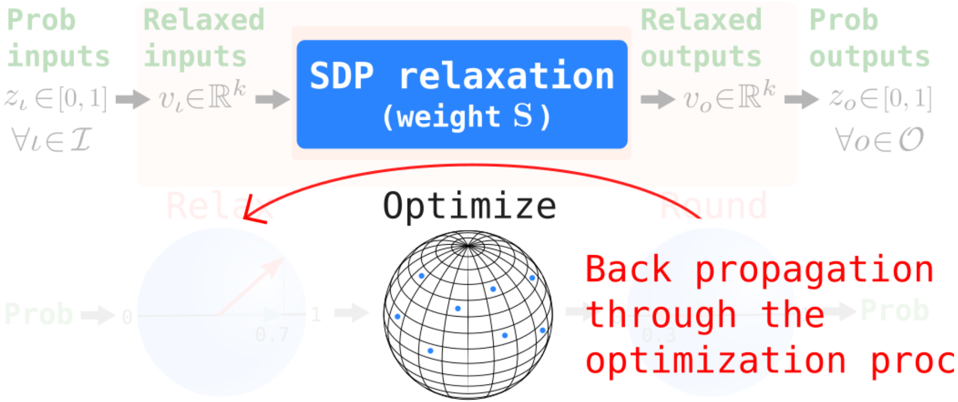
Complexity reduced from $O(n^6 \log\log \frac{1}{\epsilon})$ of interior point methods to $O(n^{1.5} m \log \frac{1}{\epsilon})$ of our method, where $m$ is #clauses.

# Differentiate through the optimization problem



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$$F_i(S, V(S)) = v_i + \text{normalize}(VS^T s_i - \|s_i\|^2 v_i) = 0, \ \forall i$$

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Thus, can apply implicit function theorem on the total derivatives

$$\frac{\partial \vec{F}(\vec{S}, \vec{V}(S))}{\partial \vec{S}} = 0 \implies \frac{\partial \vec{F}(\vec{S}, \vec{V})}{\partial \vec{S}} + \frac{\partial \vec{F}(\vec{S}, \vec{V})}{\partial \vec{V}} \cdot \frac{\partial \vec{V}}{\partial \vec{S}} = 0$$

Solve the above **linear system** of $\partial \vec{V} / \partial \vec{S}$ to backprop

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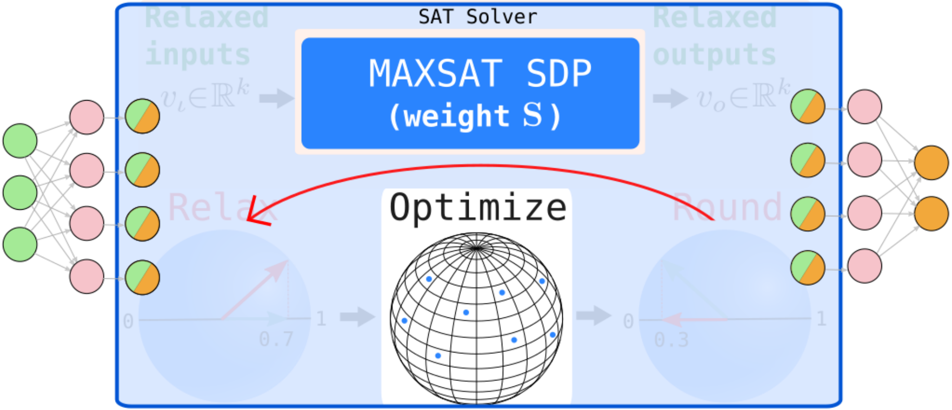

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- Only SDP with diagonal constraints, limiting representation
- Adding auxiliary variable (gadget) increases representation power

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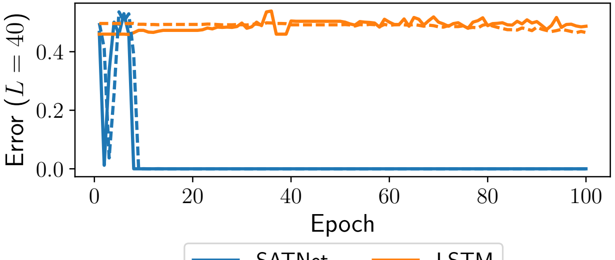
- Chained (recurrent) SATNet-based network learns parity function for up to length 40 strings from 10K examples



# Illustration: Learning Sudoku

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | 3 | | | 7 | | | | |
| 6 | | | 1 | 9 | 5 | | | |
| | 9 | 8 | | | | | 6 | |
| 8 | | | | 6 | | | | 3 |
| 4 | | | 8 | | 3 | | | 1 |
| 7 | | | | 2 | | | | 6 |
| | 6 | | | | | 2 | 8 | |
| | | | 4 | 1 | 9 | | | 5 |
| | | | | 8 | | | 7 | 9 |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 5 | 3 | 4 | 6 | 7 | 8 | 9 | 1 | 2 |
| 6 | 7 | 2 | 1 | 9 | 5 | 3 | 4 | 8 |
| 1 | 9 | 8 | 3 | 4 | 2 | 5 | 6 | 7 |
| 8 | 5 | 9 | 7 | 6 | 1 | 4 | 2 | 3 |
| 4 | 2 | 6 | 8 | 5 | 3 | 7 | 9 | 1 |
| 7 | 1 | 3 | 9 | 2 | 4 | 8 | 5 | 6 |
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- Learning 9x9 Sudoku from 9K examples
- Single SATNet layer on one-hot-encoded input puzzles

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Original Sudoku.

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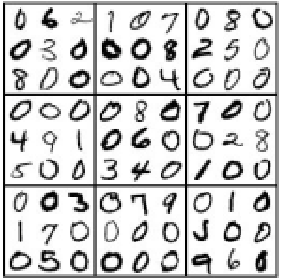
Permuted Sudoku.

# Illustration: MNIST Sudoku



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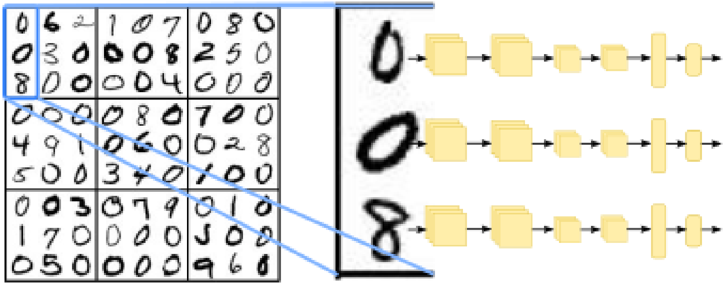

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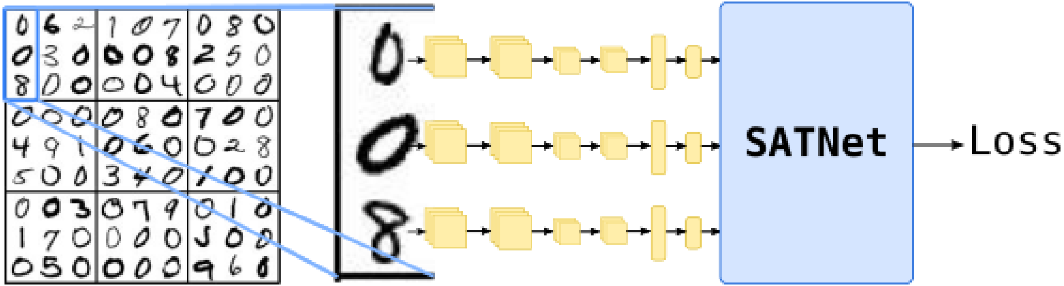

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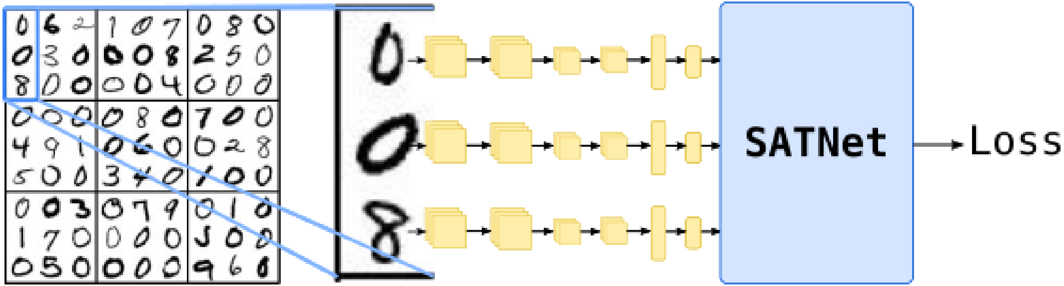


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- Getting example "correct" requires correct Sudoku solution *and* predicting all MNIST test digits correctly

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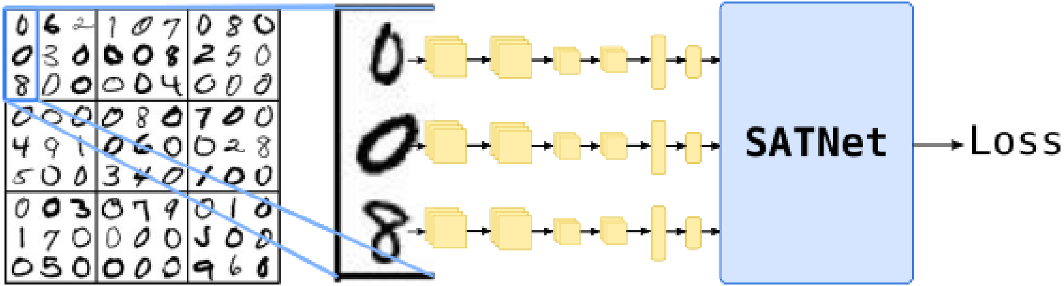


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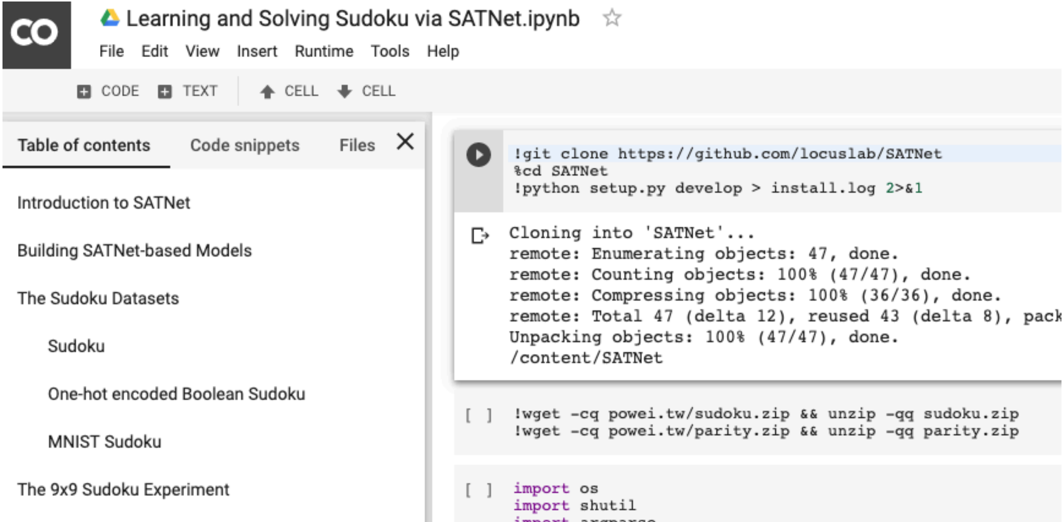
- Getting example "correct" requires correct Sudoku solution *and* predicting all MNIST test digits correctly
- $85\%$ accuracy on correct ConvNet input

# Code and Colab

Code available at `https://github.com/locuslab/SATNet`



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Possible extensions:

- Incorporating known rules into the system
- Exploiting structures of the clause matrix

Poster at Pacific Ballroom #26