Multivariate Submodular Optimization

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Joint work with Bruce Shepherd (UBC)

Some Definitions

- Ground set $V = \{1, 2, \ldots, n\}$ with power set $2^V = \{A : A \subseteq V\}$
- A set function $f : 2^V \to \mathbb{R}$ is *submodular* if $\forall A \subseteq B$ and $v \notin B$:

$$f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B)$$

- Submodularity = diminishing returns property
- f is *monotone* if $f(A) \leq f(B)$ for $A \subseteq B$

Submodularity in ML

- Sensing & Information gathering: Singh, Krause, Guestrin, Kaiser, Batalin '07
- Documment summarization: Lin and Bilmes '11
- Viral marketing: Kempe, Kleinberg, Tardos '03
- Data subset selection & Active learning: Wei, Iyer, Bilmes '15
- Robotics: Dey, Liu, Herbert, Bagnell '12
- Feature selection: Liu, Wei, Kirchhoff, Song, Bilmes '13
- Image segmentation: Kim, Xing, Fei-Fei, Kanade '11
- Diversity: Prasad, Jegelka, Batra '14

Submodular Optimization

Given a submodular function f and a family of feasible sets $\mathcal{F} \subseteq 2^V$:

Submodular Optimization Problems:

$$\text{SO}(\mathcal{F}) \quad \textit{min} \ / \ \textit{max} \ f(S) : S \in \mathcal{F}$$

where:

- $\mathcal{F} = \{S \subseteq V : |S| \leq k\}$
- $\mathcal{F} = \{S : S \subseteq V\}$
- $\mathcal{F} = \{\text{spanning trees of some graph } G\}$
- $\mathcal{F} =$ matroid or p-matroid intersection

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Multi-Agent Submodular Optimization Problems:

$$\mathrm{MASO}(\mathcal{F}) \quad \mathit{min} \ / \ \mathit{max} \ \sum_{i=1}^{k} f_i(S_i) : S_1 \uplus S_2 \uplus \cdots \uplus S_k \in \mathcal{F}$$

where $\uplus$ denotes the union of disjoint sets.

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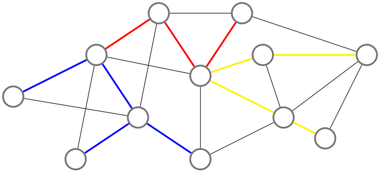

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Our Results

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Question 1: Is MVSO really more general than MASO? Yes!

$$(\textbf{MV} - \textbf{Min}) \quad \begin{array}{ll} \min & g(S_1, \ldots, S_k) \\ \text{s.t.} & S_1 \uplus \cdots \uplus S_k = V \end{array} \qquad (\textbf{MA} - \textbf{Min}) \quad \begin{array}{ll} \min & \sum_{i=1}^{k} f_i(S_i) \\ \text{s.t.} & S_1 \uplus \cdots \uplus S_k = V \end{array}$$

Theorem

There is a tight $\tilde{\Omega}(n)$ gap between the approximation factors for MV-Min and MA-Min where all the functions are nonnegative and monotone.

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Question 2: Given an α-approx for SO($\mathcal{F}$), what can be said about MVSO($\mathcal{F}$)? [We refer to the additional incurred loss as the MV gap]

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Theorem (Maximization)

- *MV gap of* $1 - 1/e$ *for monotone functions and* 0.385 *for nonmonotone*
- *MV gap of 1 for several families such as matroids and p-systems*
- *Accelerated greedy and distributed algorithms still work for MVSO*

Theorem (Minimization)

- *Essentially tight approximation factors w.r.t. the curvature of g*