Homomorphic Sensing

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Shuffled Linear Regression

$$Ax = b \longleftarrow \text{known only up to a permutation}$$

[1] A. Pananjady, M. J. Wainwright, T. D. Courtade, "Linear regression with shuffled data: statistical and computational limits of permutation recovery", IEEE Transactions on Information Theory, 2018.

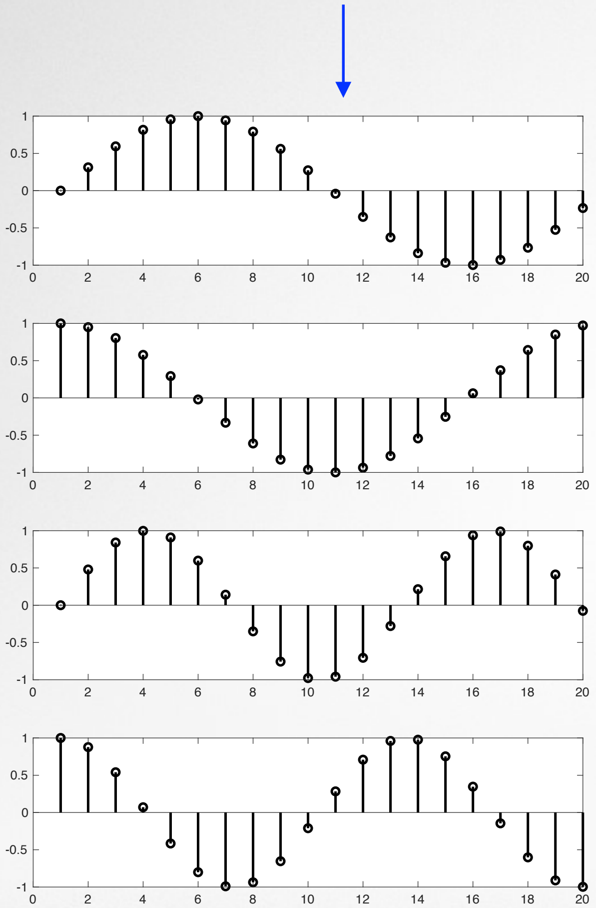
[2] J. Unikrishnan, S. Haghighatsoar, M. Vetterli, "Unlabeled sensing with random linear measurements", IEEE Transactions on Information Theory, 2018.

[3] M. Slawski and E. Ben-David, "Linear regression with sparsely permuted data", Electronic Journal of Statistics, 2019.

[4] X. Song, H. Choi, Y. Shi, "Permuted linear model for header-free communication via symmetric polynomials", ISIT, 2018.

Shuffled Linear Regression

4-dimensional signal subspace $\mathcal{V}$



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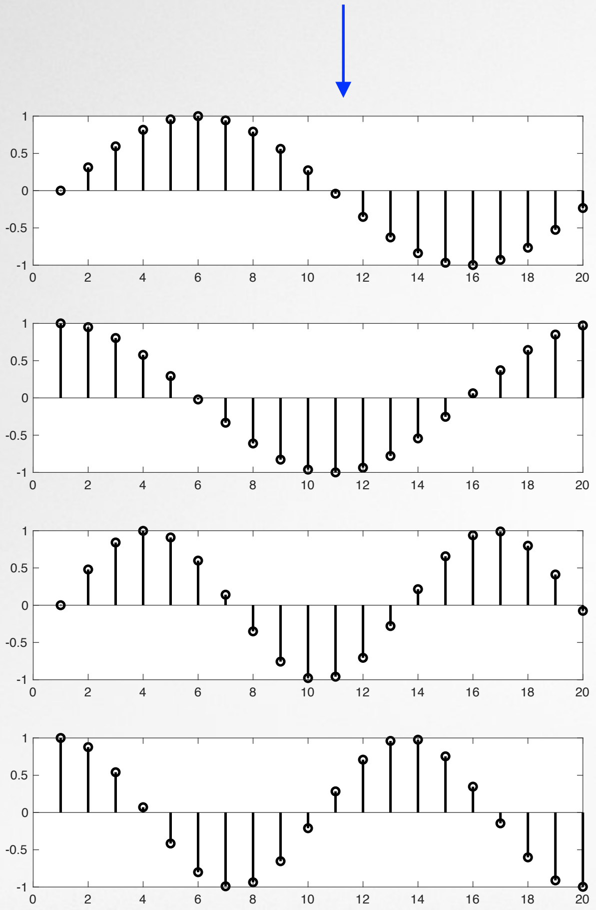

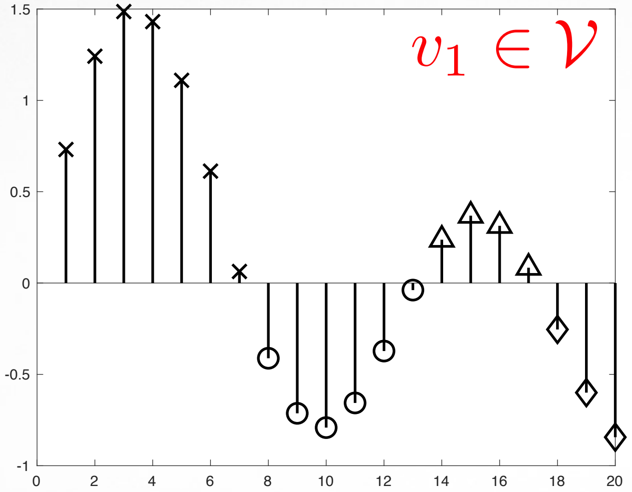

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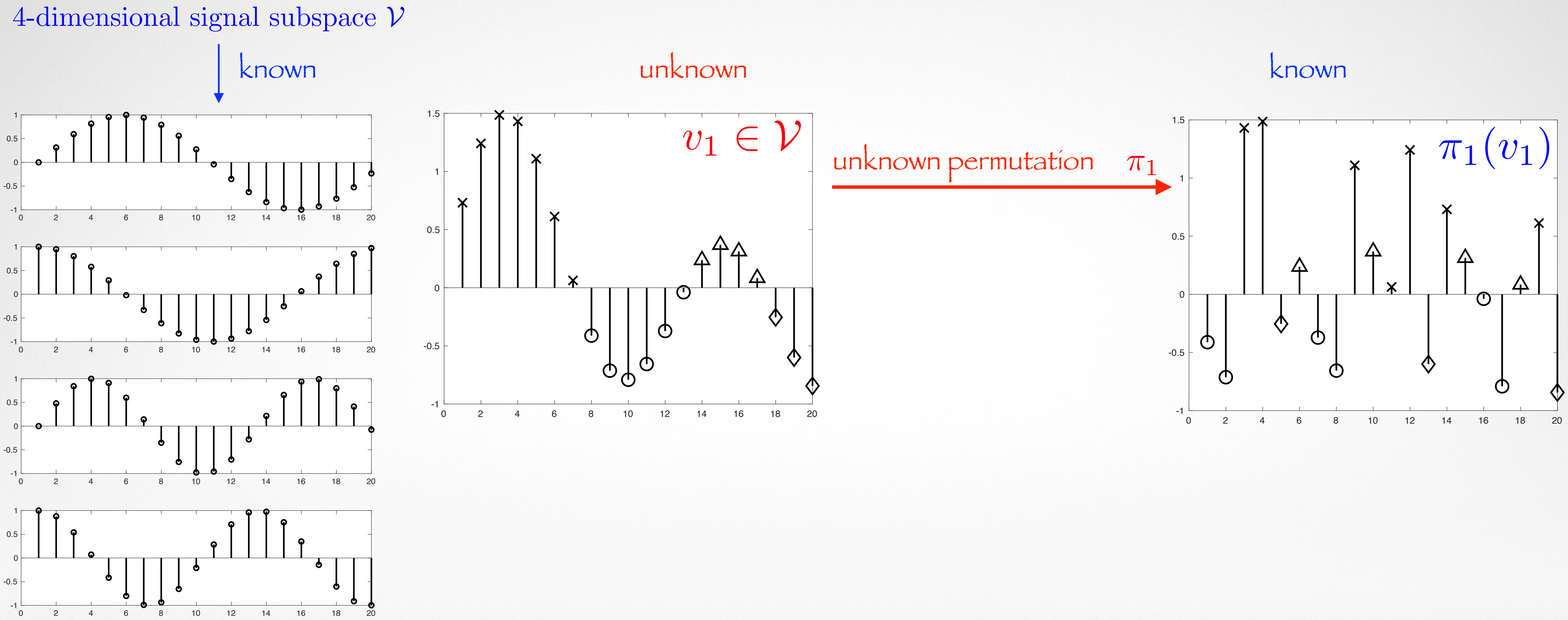
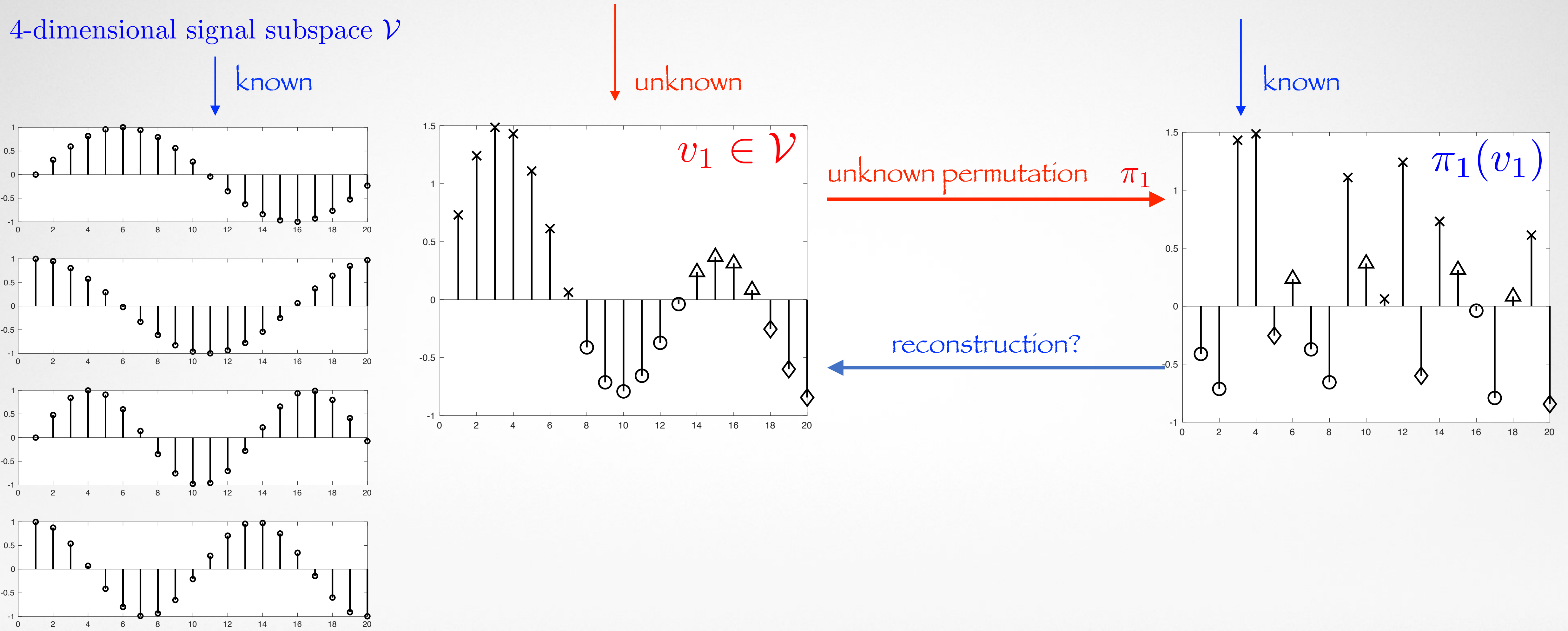


Shuffled Linear Regression

Reconstruct the original signal from its shuffled measurements.



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4-dimensional signal subspace $\mathcal{V}$

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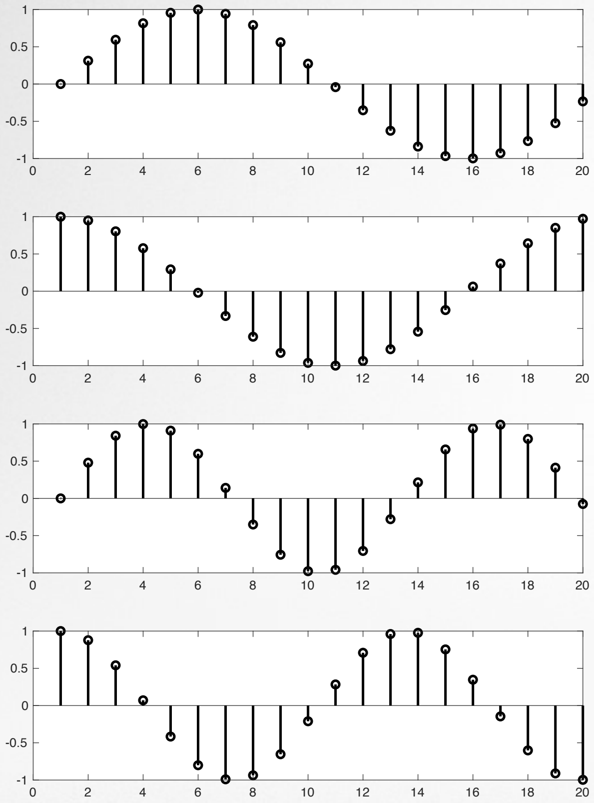
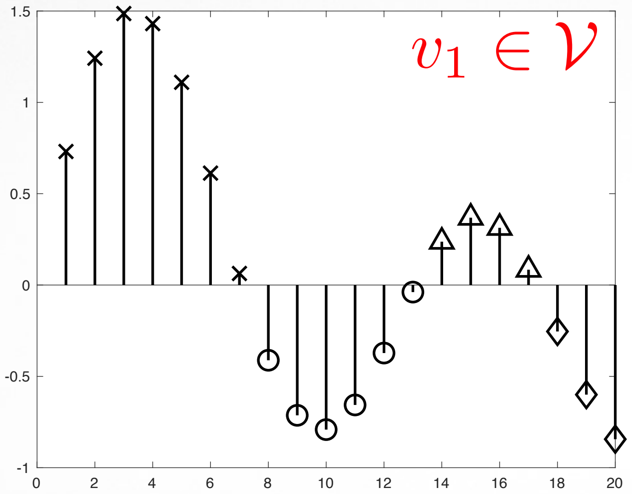
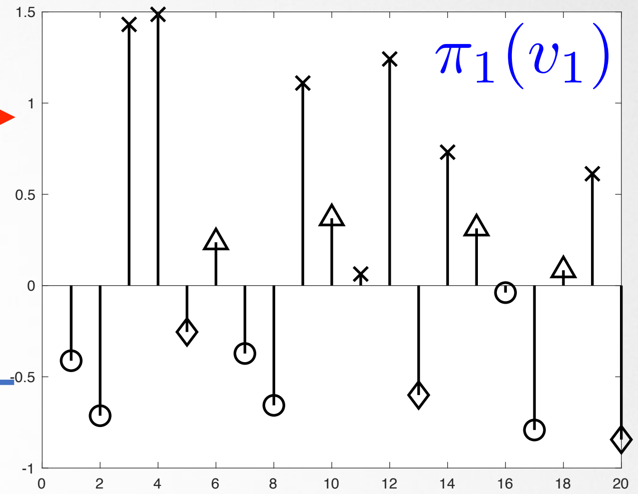




unknown permutation π_1

reconstruction?



unique reconstruction?

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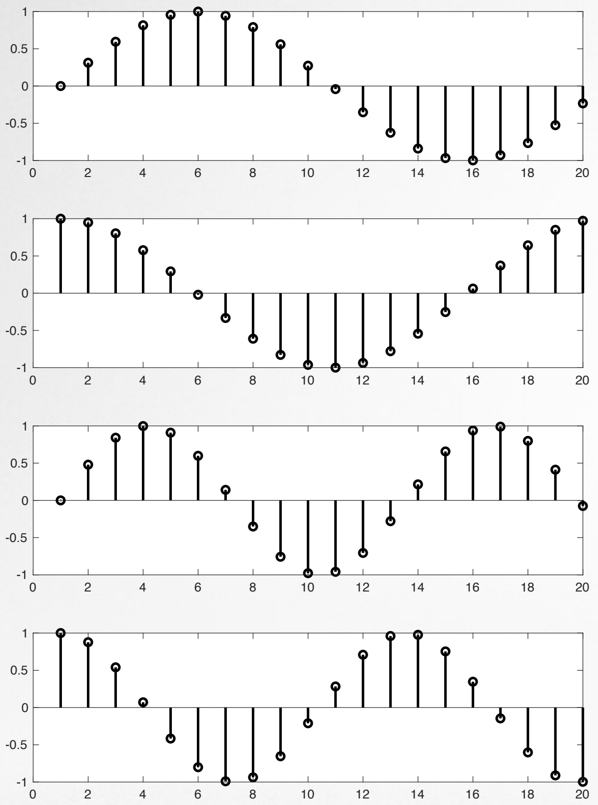

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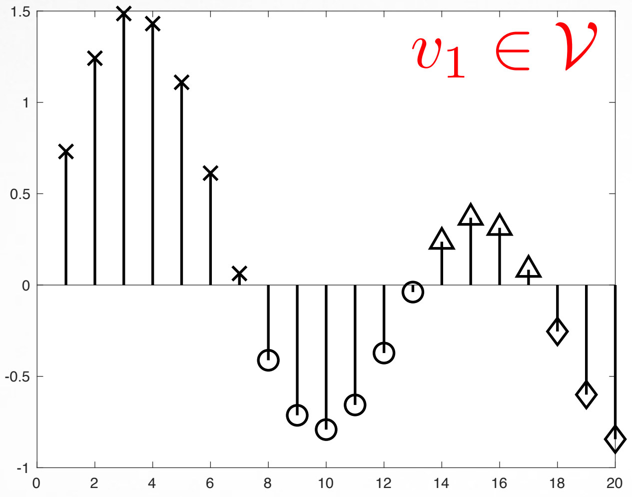

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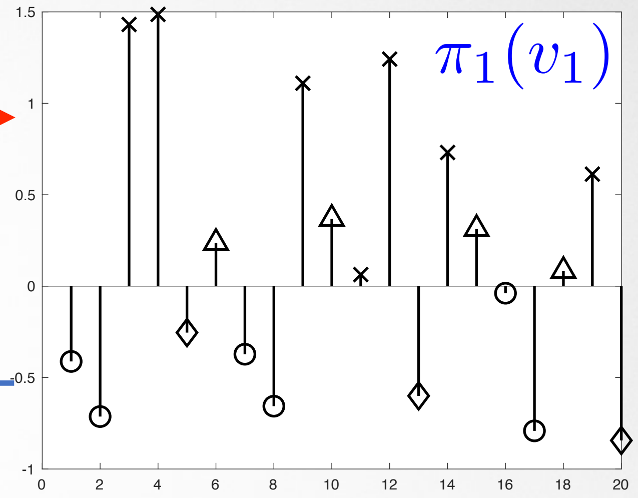
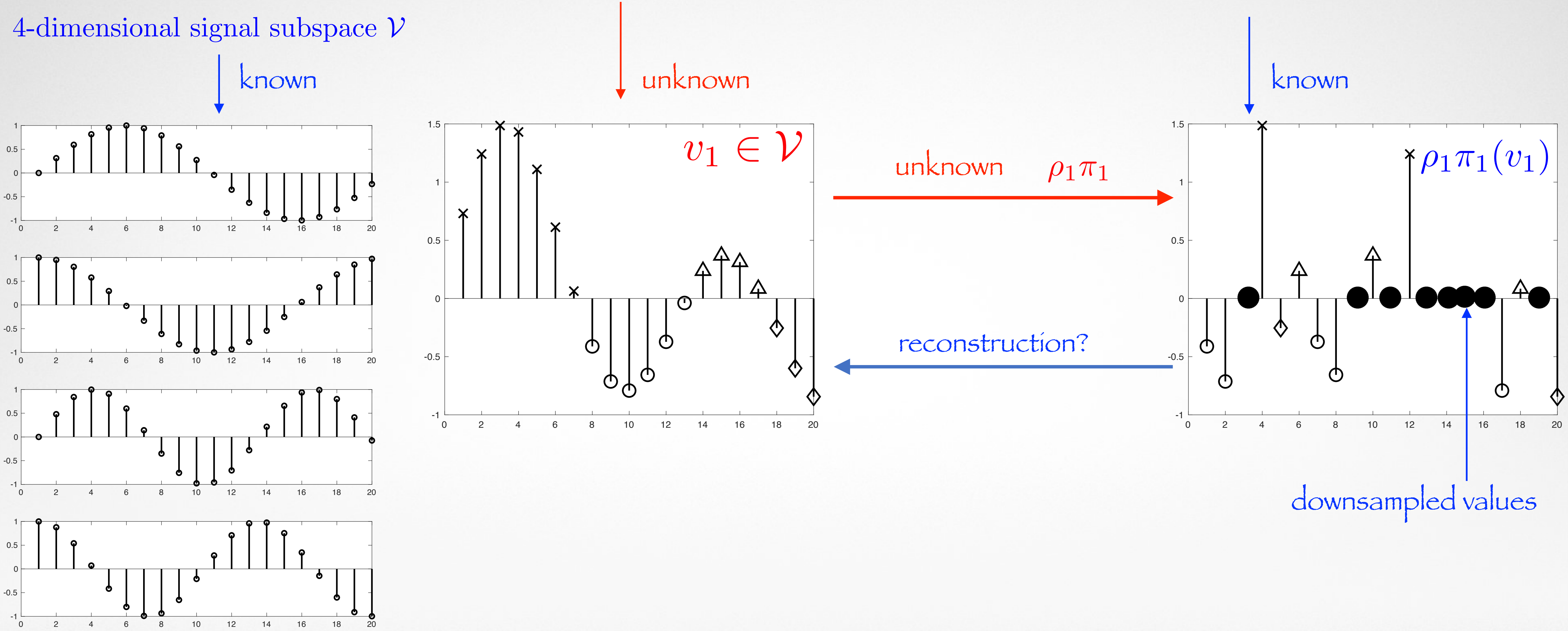


unique reconstruction: $v_1 \neq v_2 \in \mathcal{V} \;\Rightarrow\; \pi_1(v_1) \neq \pi_2(v_2)$

permutations

Unlabeled Sensing

Reconstruct the original signal from its shuffled and downsampled measurements.



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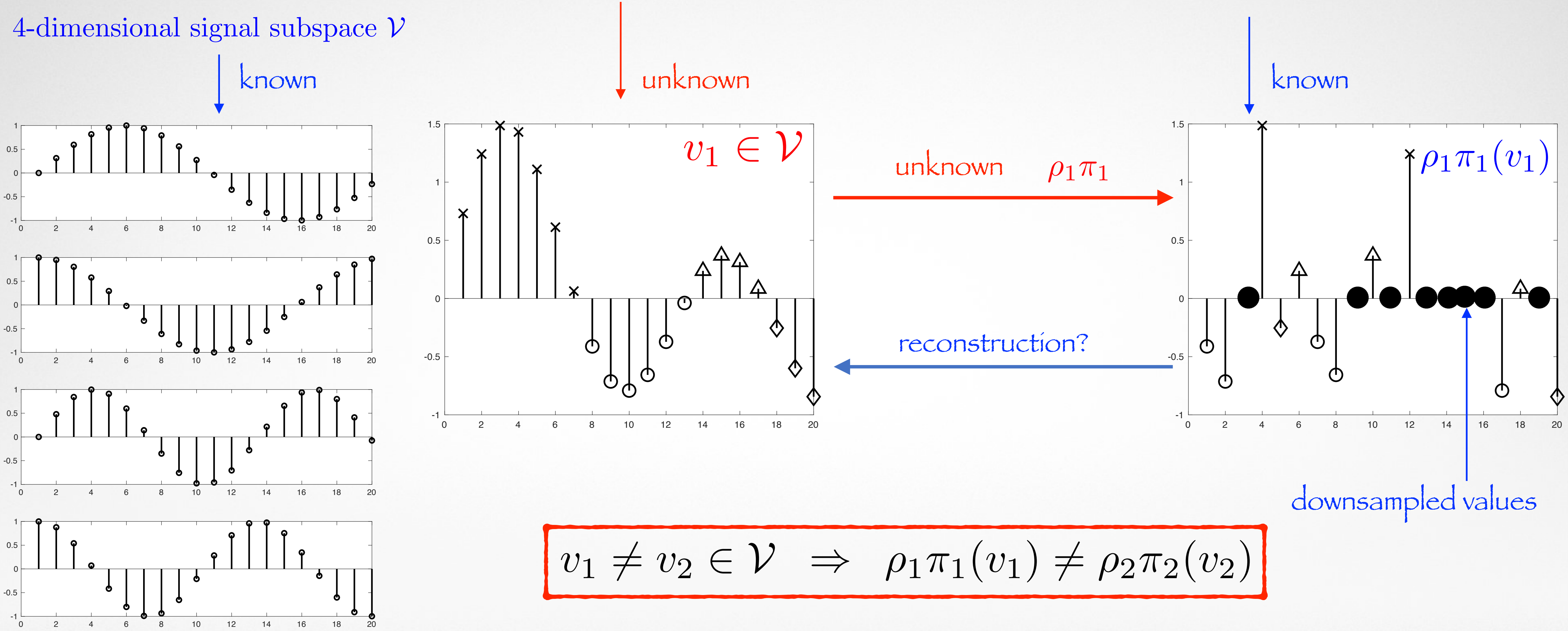

$$v_1 \neq v_2 \in \mathcal{V} \quad \Rightarrow \quad \rho_1 \pi_1(v_1) \neq \rho_2 \pi_2(v_2)$$

The Problem

$\mathcal{V}$: generic linear subspace of $\mathbb{R}^m$

$\mathcal{T}$: a finite set of linear transformations $\tau : \mathbb{R}^m \to \mathbb{R}^m$

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homomorphic sensing property **HSP**

$$v_1 \neq v_2 \in \mathcal{V} \;\Rightarrow\; \tau_1(v_1) \neq \tau_2(v_2), \;\; \forall \tau_1, \tau_2 \in \mathcal{T}$$

Main Result

$\mathcal{V}$: generic linear subspace of $\mathbb{R}^m$

τ_1, τ_2: linear transformations $\mathbb{R}^m \to \mathbb{R}^m$

$\left(\mathrm{rank}(\tau_2) \geq \mathrm{rank}(\tau_1)\right)$

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Theorem **HSP** ✔

$$\dim(\mathcal{V}) \leq \min\left\{\frac{1}{2}\mathrm{rank}(\tau_2), \mathrm{codim}(\mathcal{U})\right\}$$

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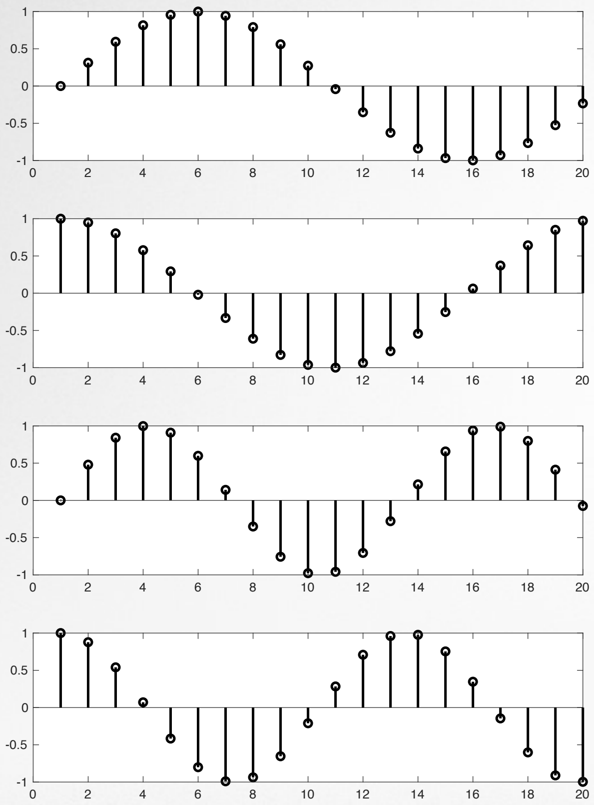

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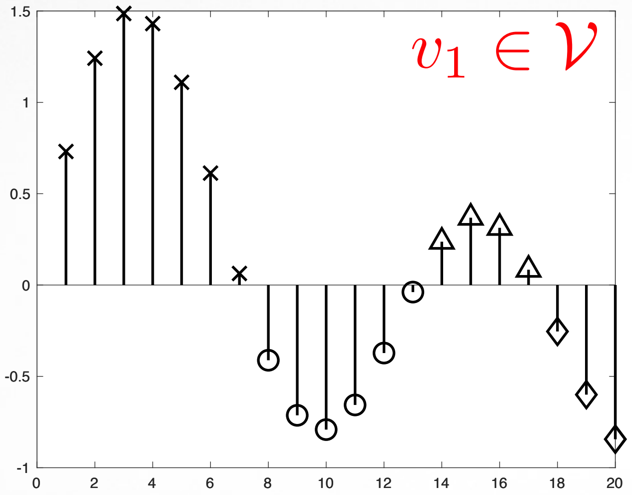

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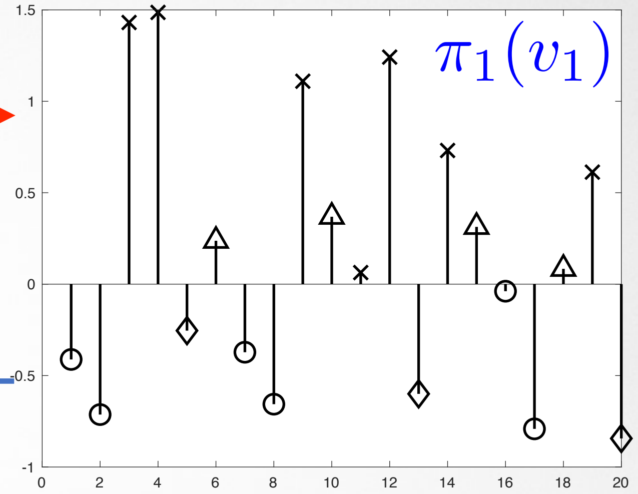

Corollary HSP

$$\dim(\mathcal{V}) \leq \frac{m}{2}$$

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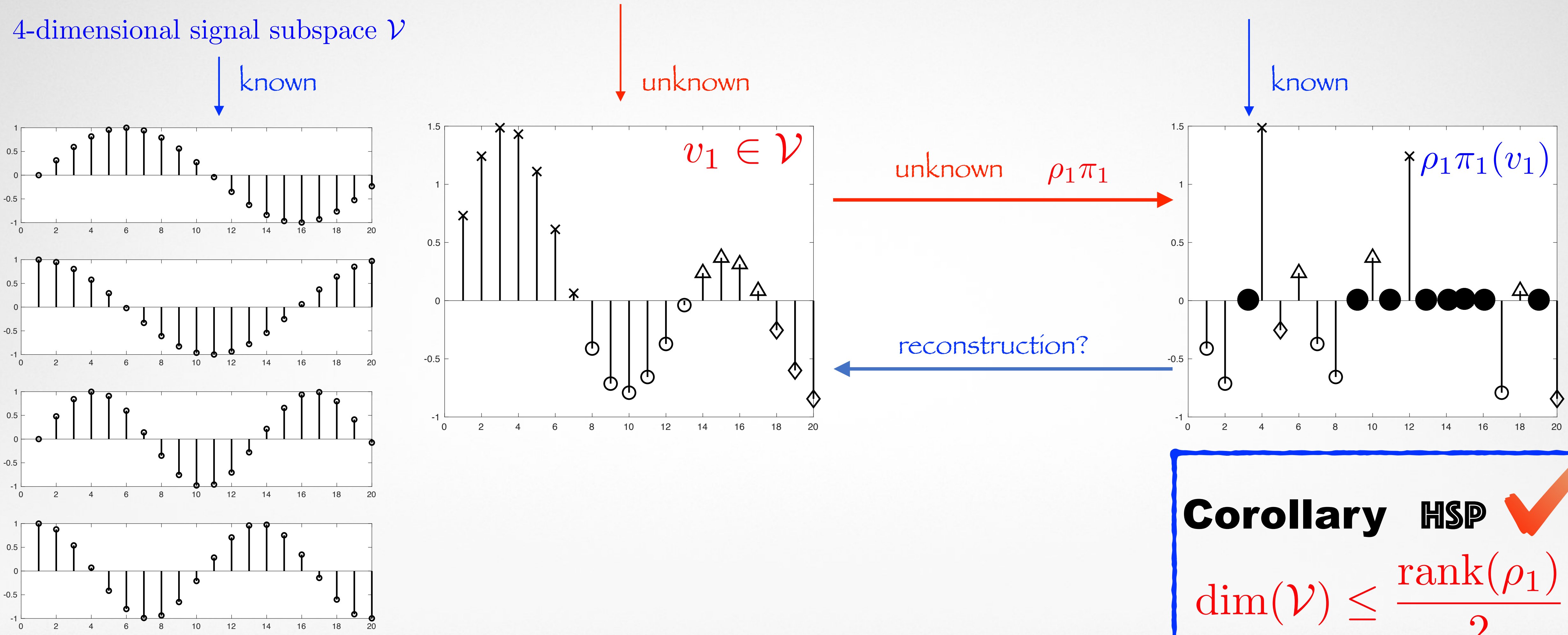

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