Inference and Sampling of K_{33}-free Ising Models

Valerii Likhosherstov[1], Yury Maximov[1,2], Michael Chertkov[1,2,3]

[1] Skolkovo Institute of Science and Technology, Moscow, Russia
[2] Theoretical Division and Center for Nonlinear Studies, Los Alamos National Laboratory, Los Alamos, NM, USA
[3] Graduate Program in Applied Mathematics, University of Arizona, Tucson, AZ, USA

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Definitions and Notations

For a graph $G = (V, E)$, $|V| = N$, **zero-field Ising model** is a distribution over $S \in \{-1, +1\}^N$ defined as

$$\mathbb{P}(S = X) = \frac{1}{Z} \exp\Big(\sum_{e=\{v,w\} \in E} J_e x_v x_w \Big) \tag{1}$$

where $\{J_e\}_{e \in E}$ are **pairwise interactions** and

$$Z(J) = \sum_{X \in \{-1,+1\}^N} \exp\Big(\sum_{e=\{v,w\} \in E} J_e x_v x_w \Big) \tag{2}$$

is a **partition function**.

Problem Overview

Question

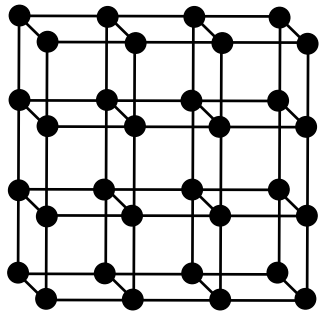
For which graphs G can we compute Z and sample from $\mathbb{P}(S)$?

Fact (Barahona, 1982)

Even when G is a two-level square grid, the task of finding Z is NP-hard.

Fact (Jerrum & Sinclair, 1993)

Even when $J > 0$, the task of finding Z is #P-complete.



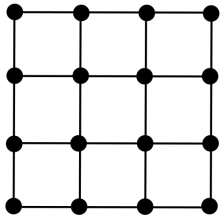
Problem Overview: Planar Zero-field Ising Models

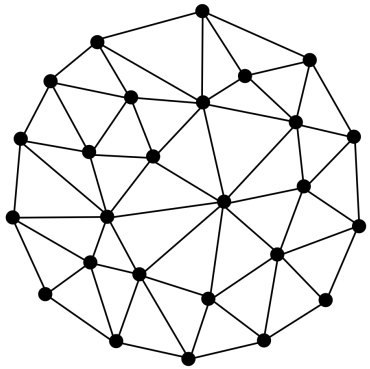
Planar zero-field Ising model - a case when G is planar.

Theorem

Given a planar zero-field Ising model, finding Z and sampling from $\mathbb{P}(S)$ takes $O(N^{\frac{3}{2}})$ time.

- Theorem is due to (Kasteleyn, 1963; Wilson, 1997; Schraudolph & Kamenetsky, 2009; Thomas & Middleton, 2009; 2013).
- No self-contained description of the algorithm.
- Extension to arbitrary genus g with a factor of 4^g (Gallucio & Loebl, 1999).





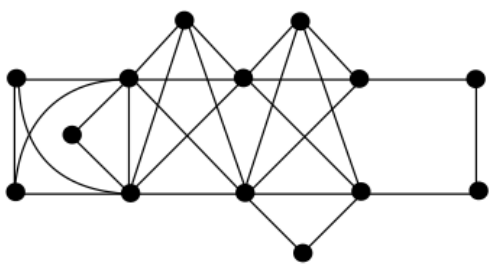
Algorithm Overview: Graph Decomposition

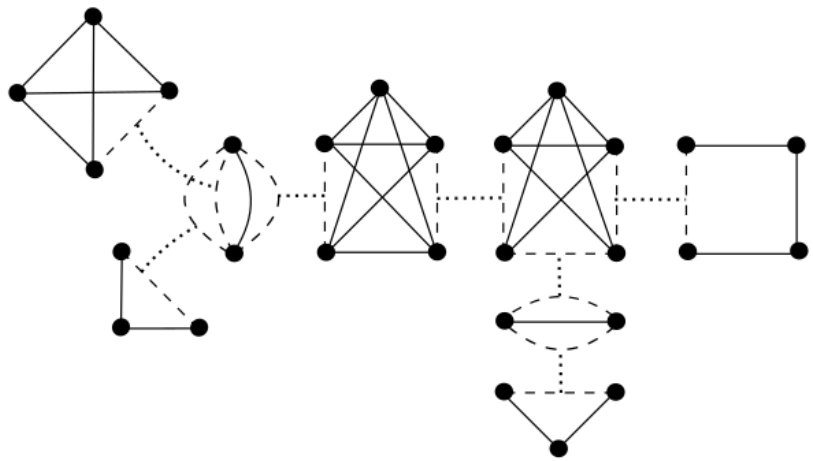
Informal definition

A tree of triconnected components T of graph G is a tree decomposition of G into **triconnected** graphs G_t with shared edges.

Theorem (Hopcroft & Tarjan, 1973)

A tree of triconnected components is unique and can be obtained in $O(N + |E|)$.





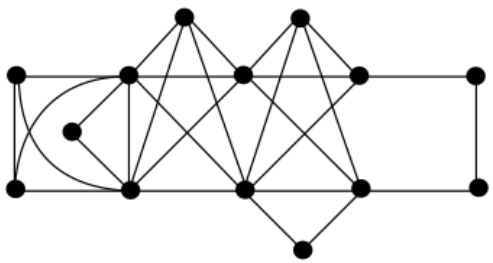
Algorithm Overview: Inference of K_{33}-free Zero-field Ising Models

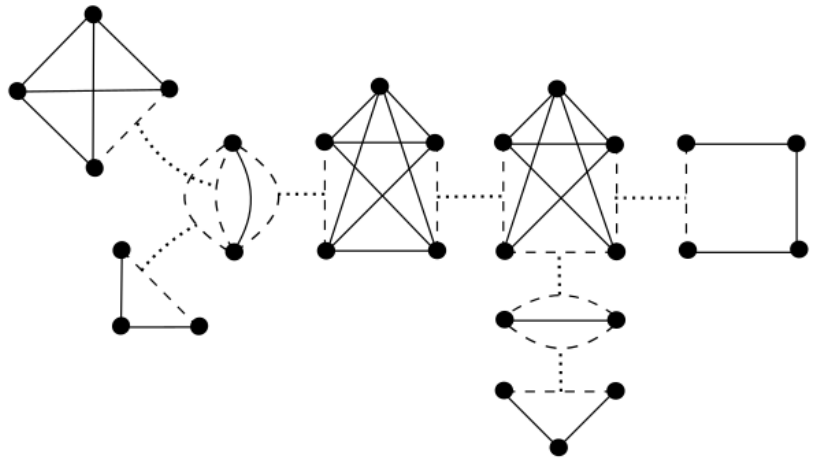
Lemma (Hall, 1943)

Graph G is K_{33}-free if and only if its triconnected components are either planar or K_5.

Theorem

Given a K_{33}-free zero-field Ising model, finding Z and sampling from $\mathbb{P}(S)$ takes $O(N^{\frac{3}{2}})$ time.





Conclusions

Main results:

- Self-contained description of $O(N^{\frac{3}{2}})$ inference and sampling of planar zero-field Ising models.
- $O(N^{\frac{3}{2}})$ inference and sampling of K_{33}-free Ising models.
- Implementation of the algorithm `https://github.com/ValeryTyumen/planar_ising`.

Poster: "Inference and Sampling of K_{33}-free Ising Models", Valerii Likhosherstov, Yury Maximov, Michael Chertkov.

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