Geometry & Symmetry in Short-and-Sparse Deconvolution

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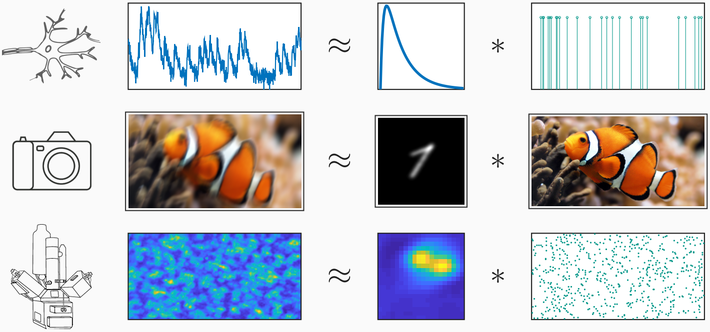
Short-and-Sparse (SaS) Deconvolution

SIGNALS CONTAINING **SHORT** **REPEATED** MOTIFS:

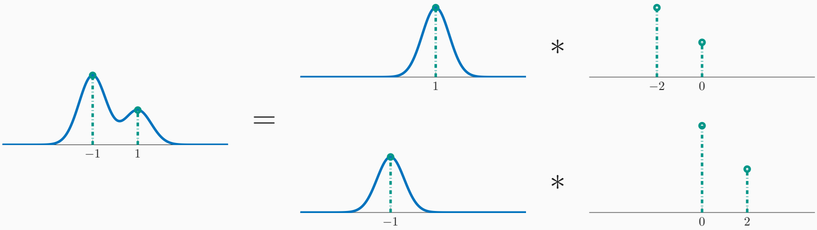


$$y \approx a_0 * x_0$$

SASD: FIND **SHORT** a_0 & **SPASE** x_0 FROM **CONVOLUTION** $y = a_0 * x_0$

Symmetric Solutions in SaSD

ALL SCALED & SHIFTS OF $(\boldsymbol{a}_0, \boldsymbol{x}_0)$ ARE SOLUTIONS



$$\boldsymbol{y} = \alpha\, s_\ell[\boldsymbol{a}_0] * (1/\alpha)\, s_{-\ell}[\boldsymbol{x}_0]$$

To solve $\boldsymbol{a}_0$...

- Fix scale $\|\widehat{\boldsymbol{a}}\|_2 = 1$
- Accept every signed shift $\widehat{\boldsymbol{a}} = \pm s_\ell[\boldsymbol{a}_0]$ as solution

Algorithm: Approximate Bilinear Lasso

NATURAL, EFFECTIVE ALGORITHM: BILINEAR LASSO

$$\min_{\boldsymbol{a}\in\mathbb{S}^{p-1},\, \boldsymbol{x}\in\mathbb{R}^n} \lambda \|\boldsymbol{x}\|_1 + \frac{1}{2}\|\boldsymbol{a}*\boldsymbol{x}-\boldsymbol{y}\|_2^2$$

THEORY: STUDY APPROXIMATE BILINEAR LASSO

$$\min_{\boldsymbol{a}\in\mathbb{S}^{p-1}} \left(\min_{\boldsymbol{x}\in\mathbb{R}^n} \lambda\boldsymbol{\rho}(\boldsymbol{x}) + \frac{1}{2}\|\boldsymbol{x}\|_2^2 + \langle \boldsymbol{a}*\boldsymbol{x}, \boldsymbol{y}\rangle \right)$$

$$=: \boxed{\min_{\boldsymbol{a}} \quad \varphi_{\mathrm{ABL}}(\boldsymbol{a}) \quad s.t. \quad \boldsymbol{a}\in\mathbb{S}^{p-1}}$$

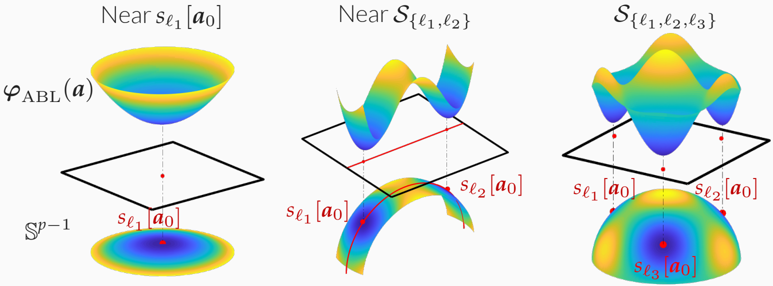
here, $\boldsymbol{\rho}$ is smoothed ℓ^1 function

$\mathbb{S}^{p-1}$ is p-dimensional sphere

Geometry of Approximate Bilinear Lasso

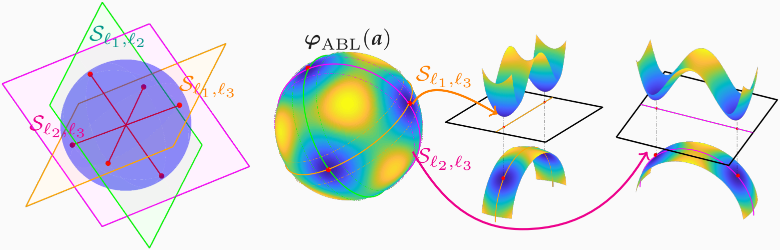
OVER **SUBSPACE $\mathcal{S}_{\{\ell_1,\dots,\ell_3\}}$ SPANNED BY SHIFTS**:

- **LOCAL MINIMIZERS** ARE NEAR SHIFTS
- **NEGATIVE CURVATURE** BREAKS SYMMETRY BETWEEN SHIFTS



Geometry of Approximate Bilinear Lasso

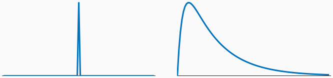
GEOMETRY OF φ_{ABL} IS **BENIGN** OVER **UNION OF SUBSPACES**



When is SaSD Easy?

SHIFT–COHERENCE μ OF $\boldsymbol{a}_0$:

$$\mu \quad = \quad \max_{i \neq j} \left| \left\langle s_i[\boldsymbol{a}_0]\,,\, s_j[\boldsymbol{a}_0] \right\rangle \right|$$



$\mu \downarrow$ $\mu \uparrow$

SPARSITY θ OF $\boldsymbol{x}_0$:

$$\boldsymbol{x}_0 \quad \sim \quad \text{Bernoulli–Gaussian}(\theta)$$

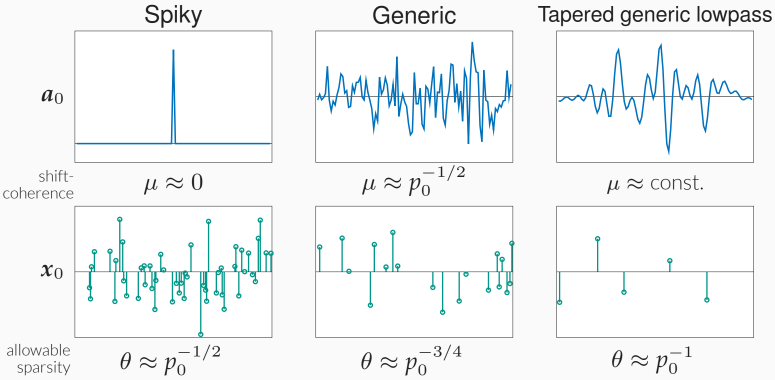
$\theta \downarrow$ $\theta \uparrow$

SASD IS **HARDER** IF...

- COHERENCE $\mu \uparrow$ (solutions closer on sphere)
- SPARSITY $\theta \uparrow$ (more unknowns)

When is SaSD Easy?

SPARSITY–COHERENCE TRADEOFF:



If μ of $\boldsymbol{a}_0$ increases from $0 \nearrow 1$, then θ of $\boldsymbol{x}_0$ decreases from $\frac{1}{\sqrt{p_0}} \searrow \frac{1}{p_0}$

Theory: Geometry & Algorithm

Thm1: Geometry of φ_{ABL} over subspaces

Given $\boldsymbol{a}_0 \in \mathbb{R}^{p_0}$, μ-shift coherent; $\boldsymbol{x}_0 \sim \mathrm{BG}(\theta)$ long and

$$\frac{1}{p_0} \lessapprox \theta \lessapprox \frac{1}{p_0\sqrt{\mu} + \sqrt{p_0}},$$

thens local minima of φ_{ABL} over UoS are close to shifts.

Thm2: Provable algorithm for SaSD

A minimizing algorithm starts and stays near a subspace, solves SaSD exactly up to a signed shift in poly time.

Wrapping Up

Main theoretical results: **geometry of objective landscape**, and a **provable algorithm** for SaSD.

Optimizing φ_{ABL} is not recommended in practice.

Algorithmic ideas (sphere, initialization, etc.) are **useful for practical method** such as bilinear Lasso.

THANK YOU!

...AND COLUMBIA UNIVERSITY IN THE CITY OF NEW YORK NSF

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