Lexicographic and Depth-Sensitive Margins in Homogeneous and Non-Homogeneous Deep Models

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Motivation

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Main Goal

We would like to understand the minima selection process in training deep neural networks.

Setting

- Empirical loss:

$$\min_{\boldsymbol{\theta}} \mathcal{L}(\boldsymbol{\theta}) = \sum_{n=1}^{N} e^{-f_n(\boldsymbol{\theta})}$$

$f_n(\boldsymbol{\theta})$ - the prediction function, N - number of samples.

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- We examine overparameterized realizable problems i.e., where it is possible to perfectly classify the training data.
- The <u>inductive bias</u> introduced in our learning process affects which specific global minimizer is chosen.

Inductive Bias Sources

1) **Regularization path:**

$$\Theta_r(\lambda) = \arg\min_{\boldsymbol{\theta}} \mathcal{L}(\boldsymbol{\theta}) + \lambda \|\boldsymbol{\theta}\|_2^2 \qquad (1)$$

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Previous Results

- For linear prediction functions:
  - Optimization path $\Rightarrow$ Max-Margin solution.

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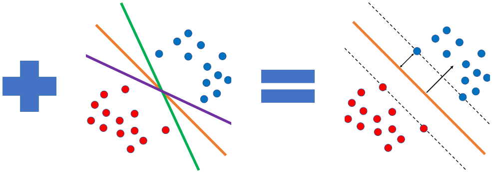

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Previous Results

- For linear prediction functions:
  - Optimization path $\Rightarrow$ Max-Margin solution.
  - Regularization and Constrained paths $\Rightarrow$ Max-Margin solution.
- For homogeneous prediction functions, e.g., ReLU networks:
  - Regularization path $\Rightarrow$ Max-Margin solution.

Soudry et al. (2018), Gunasekar et al. (2018), Rosset et al. (2004), Wei et al. (2018).

We study how infinitesimal regularization or gradient descent optimization lead to margin maximizing solutions in both homogeneous and non-homogeneous models.

Main Contributions - Non-Homogeneous Models

- For $f_n(\boldsymbol{\theta}) =$ sum of homogeneous functions of different orders: we characterized the constrained path asymptotic solution.

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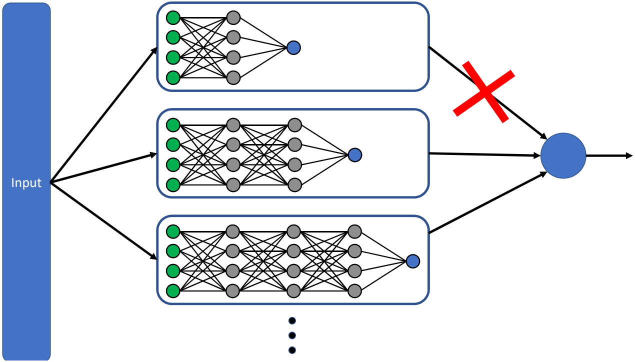
Implication

In an ensemble of homogeneous neural networks, e.g., feedforward ReLU networks, the ensemble will aim to discard the most shallow network.



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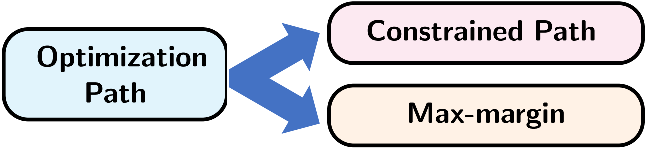
A: Yes, we find general conditions under which the optimization path converges to:

1) stationary points of the constrained path.
2) max-margin solutions.



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  - We show that the constrained path converges to a specific type of max-margin solution.

Q: Is margin maximization all that we do?

A: No. After maximizing the distance to the closest data point (max-margin), we also maximize the distance to the second closest data point, and so on.

Thank You!

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