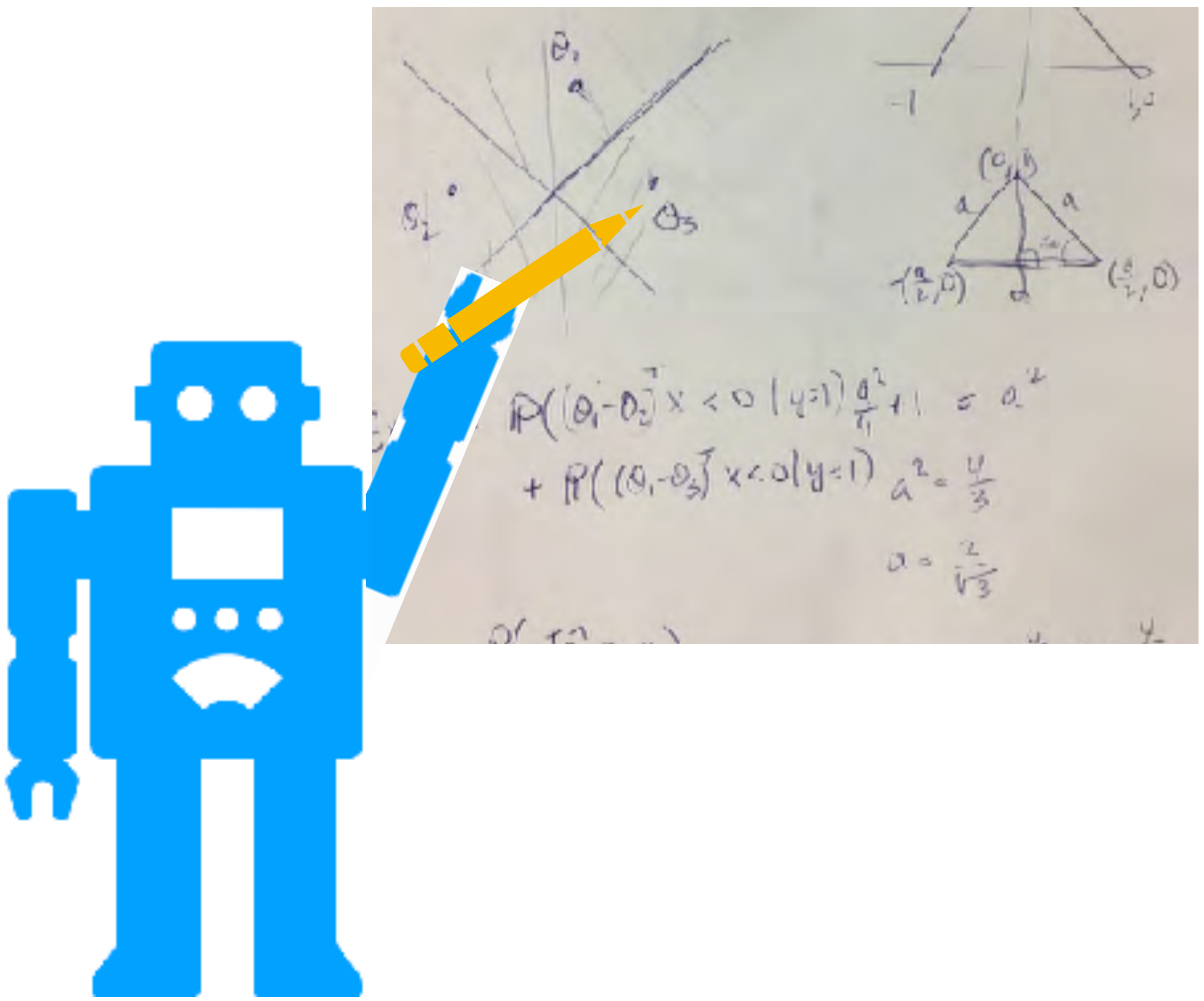
Active Learning from Theory to Practice



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ICML | 2019

Thirty-sixth International Conference on Machine Learning

Tutorial Outline

Part 1: Introduction to Active Learning (Rob)

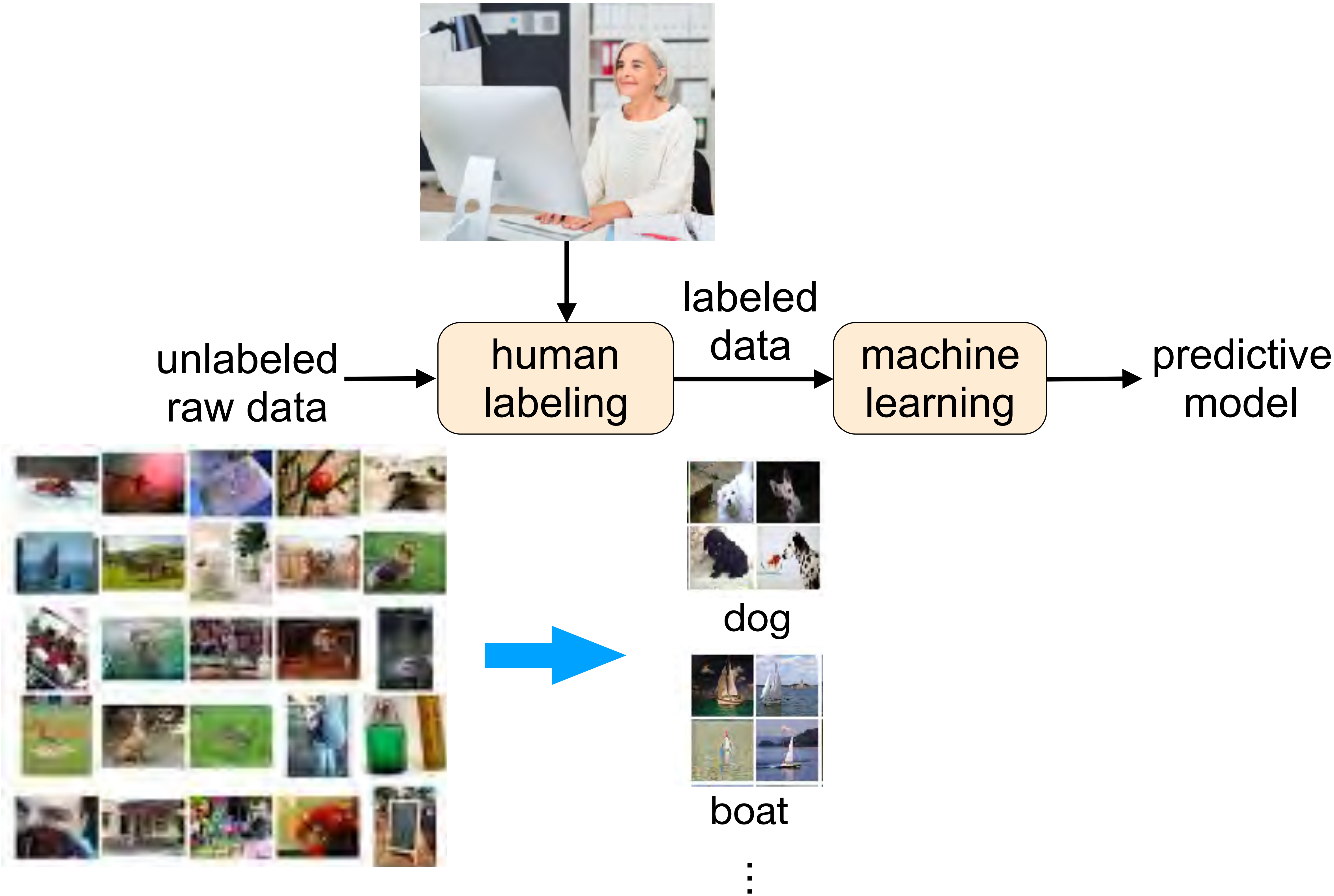
Part 2: Theory of Active Learning (Steve)

Part 3: Advanced Topics and Open Problems (Steve)

Part 4: Nonparametric Active Learning (Rob)

slides: http://nowak.ece.wisc.edu/ActiveML.html

Conventional (Passive) Machine Learning



ALL SYSTEMS GO ?

theguardian

Computers now better than humans at recognising and sorting images

millions of labeled images
1000's of human hours

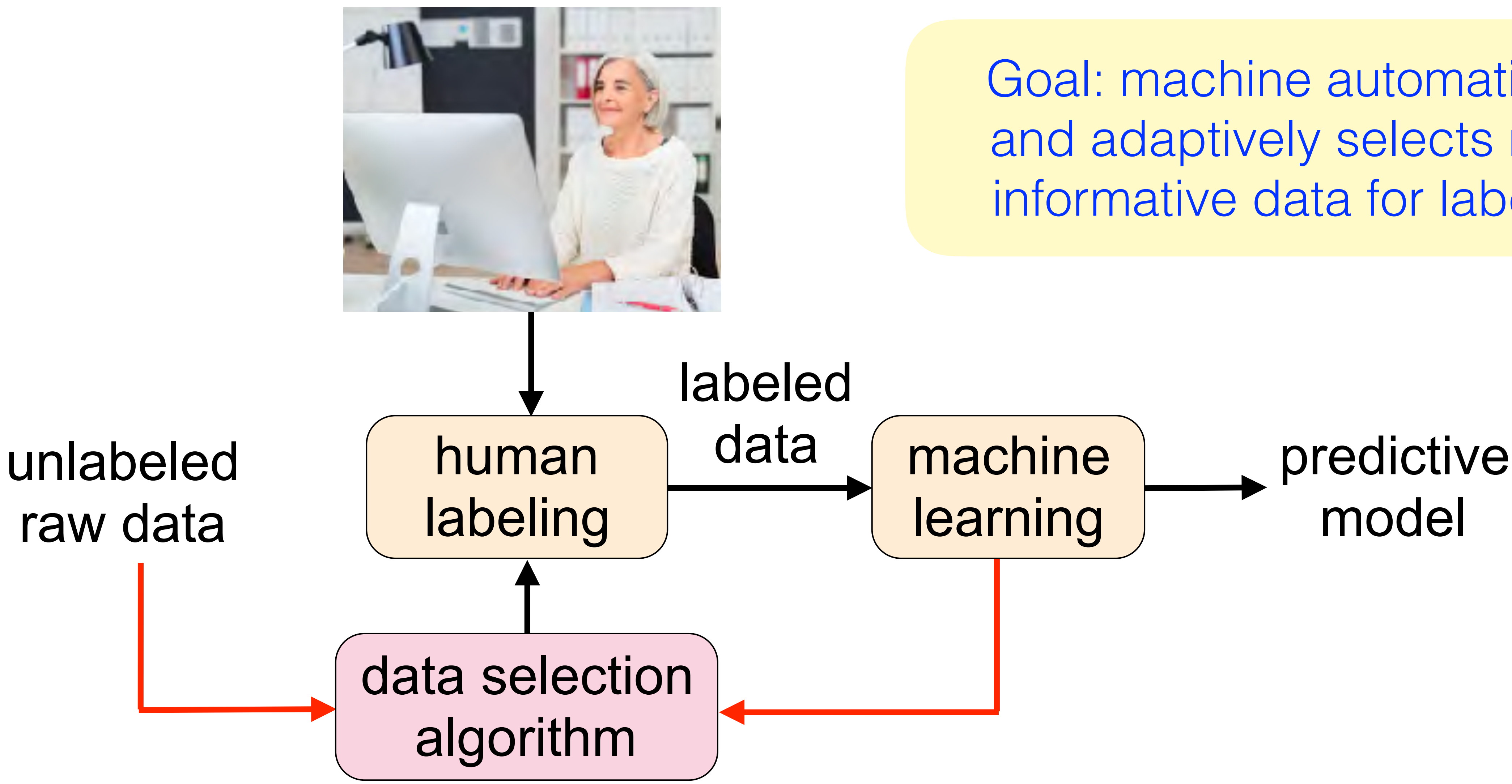
QUARTZ

Google says its new AI-powered translation tool scores nearly identically to human translators

trained on more texts than a human could read in a lifetime

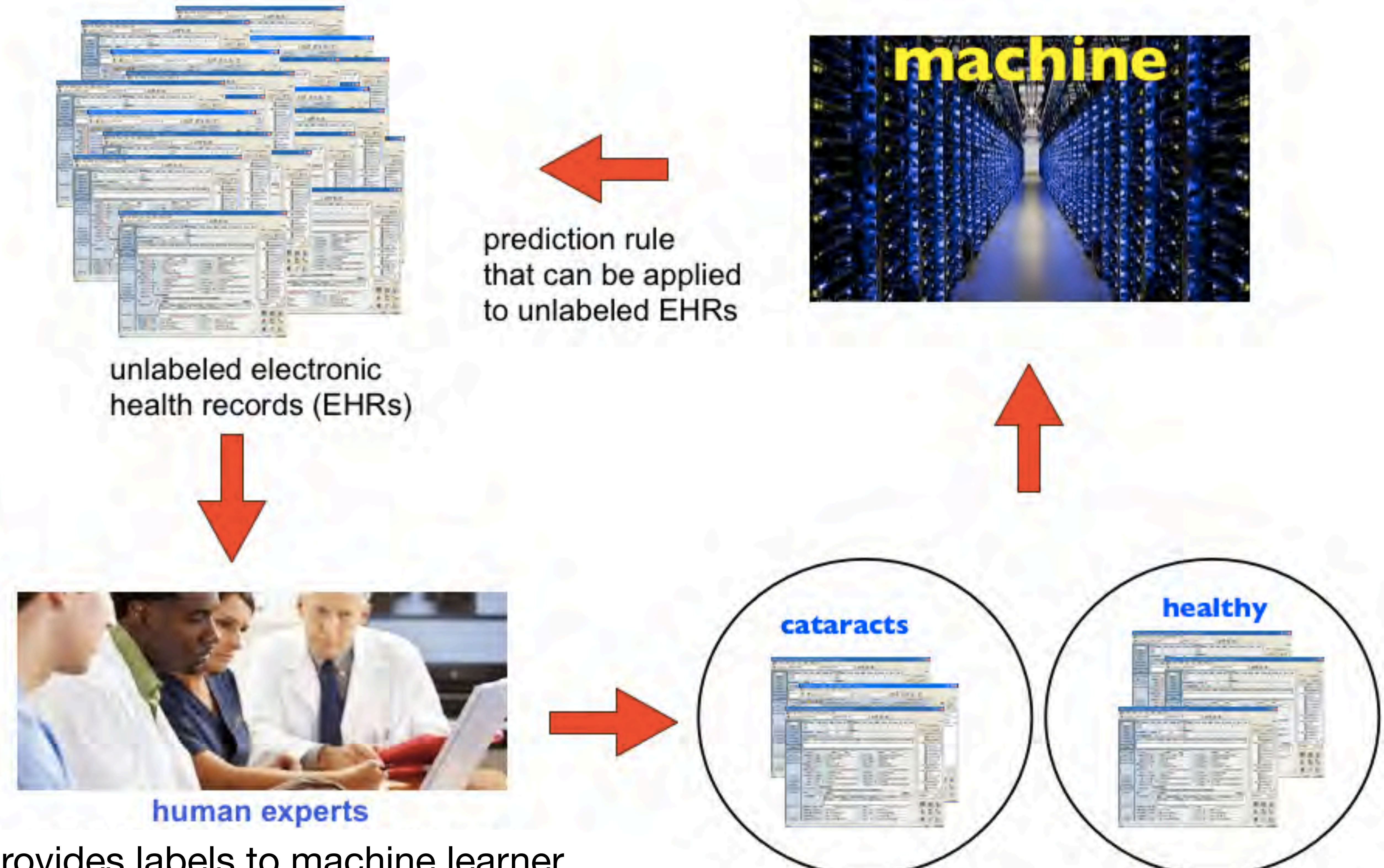
Can we train machines with less labeled data and less human supervision?

Active Machine Learning



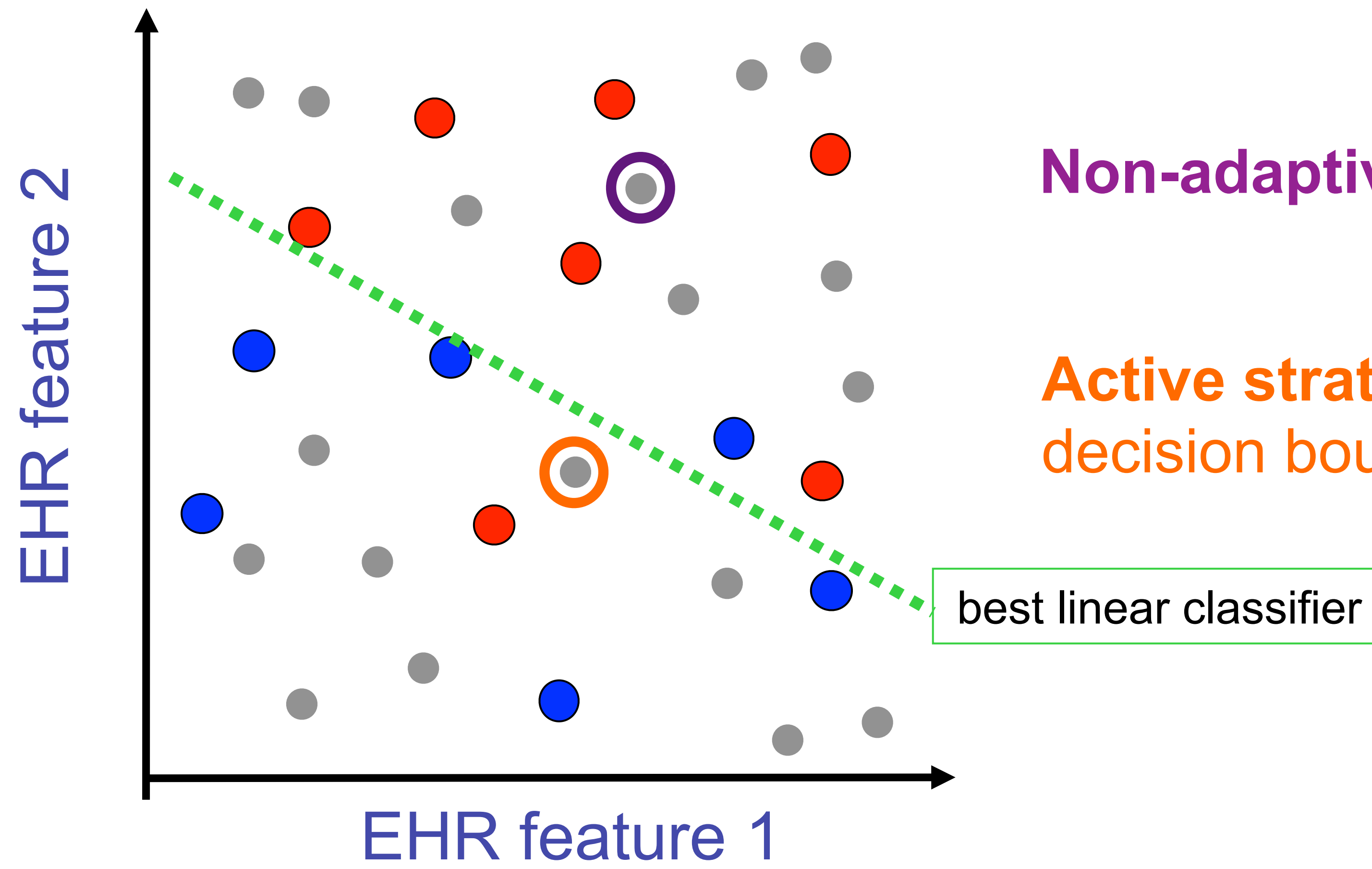
Goal: machine automatically and adaptively selects most informative data for labeling

Motivating Application



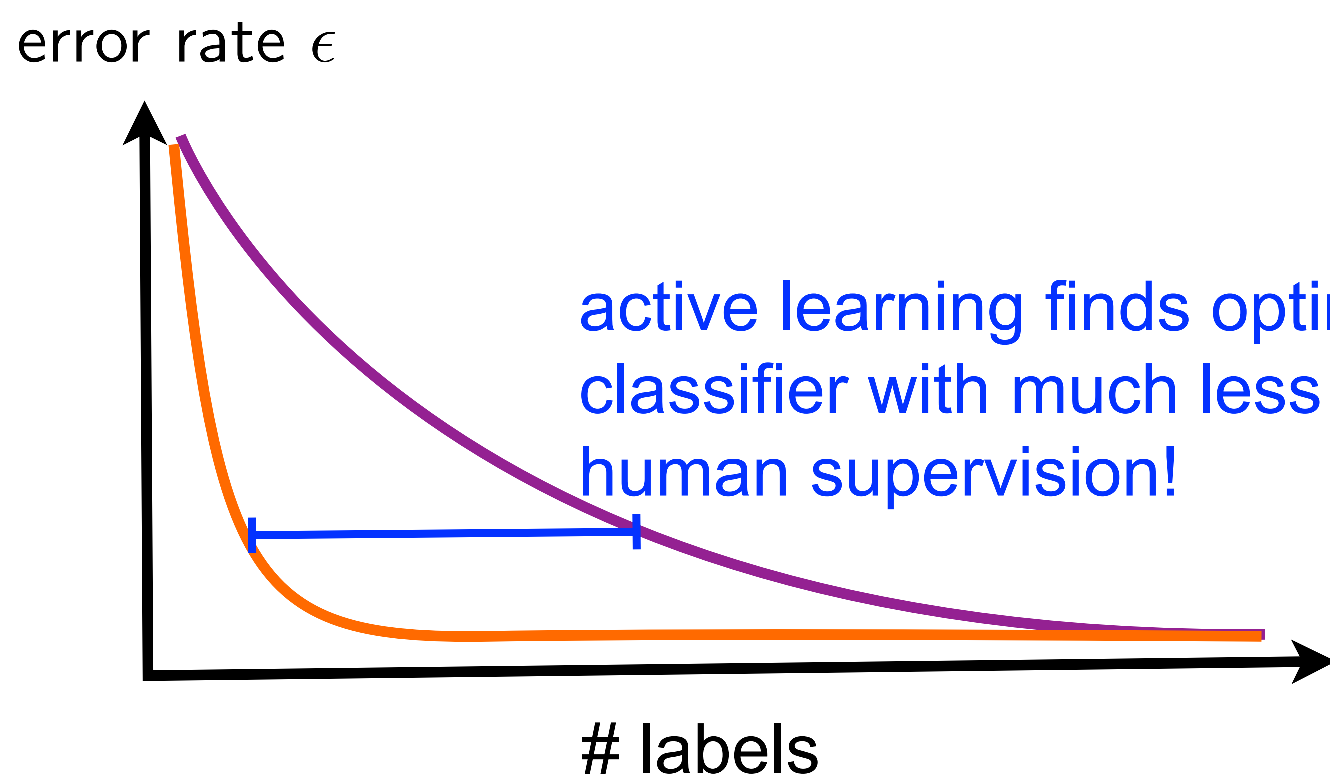
provides labels to machine learner
(several minutes / EHR)

Active Learning



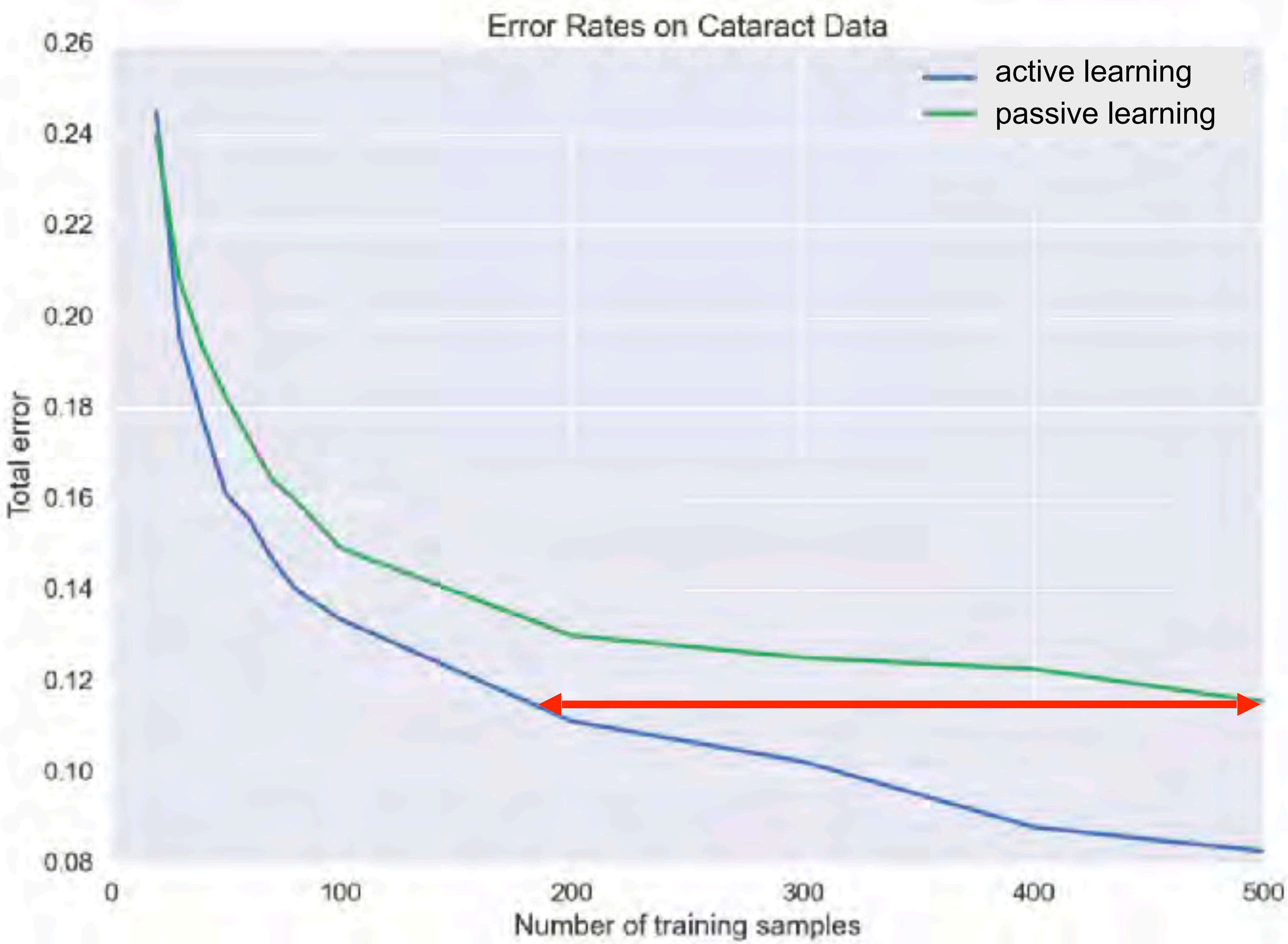
Non-adaptive strategy: Label a random sample

Active strategy: Label a sample near best decision boundary based on labels seen so far



active learning finds optimal classifier with much less human supervision!

Active Logistic Regression



11000 patient records

8000 positive
3000 negative

6182 Numerical Features

icd9 codes
lab tests
patient data

Classification task:
cataracts or healthy

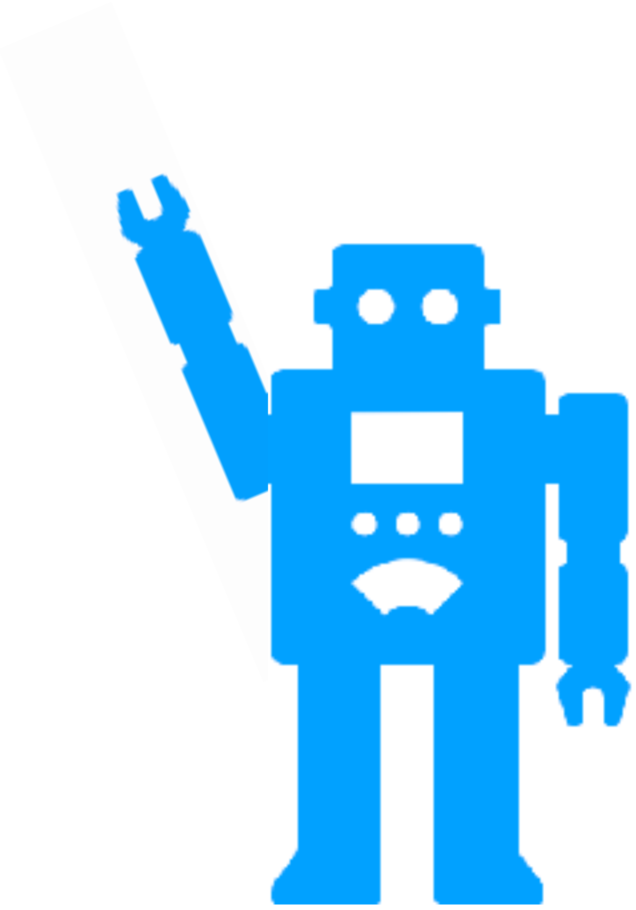
less than half as many labeled examples needed by active learning


nextml.org
NEXT
ASK BETTER QUESTIONS.
GET BETTER RESULTS.
FASTER. AUTOMATED.
GitHub
Paper
Docs
Blog
Team
Data

Active learning to optimize crowdsourcing and rating in New Yorker Cartoon Caption Contest

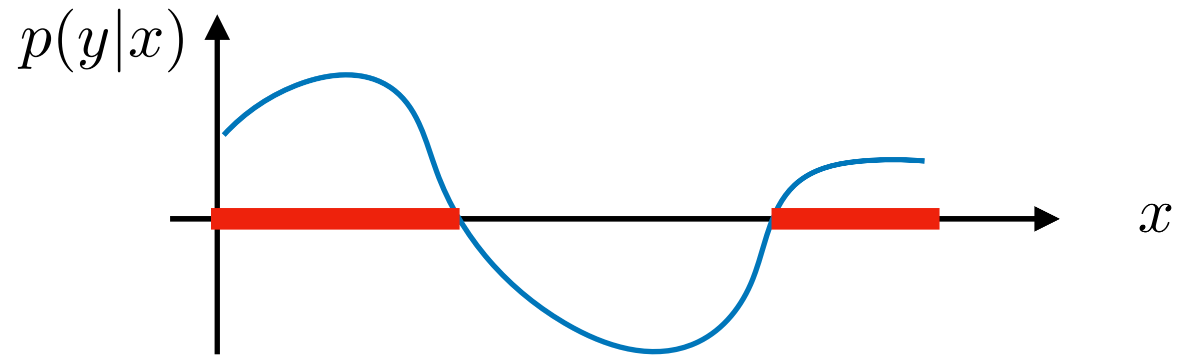
Actively learning user’s beer preferences

Principles of Active Learning

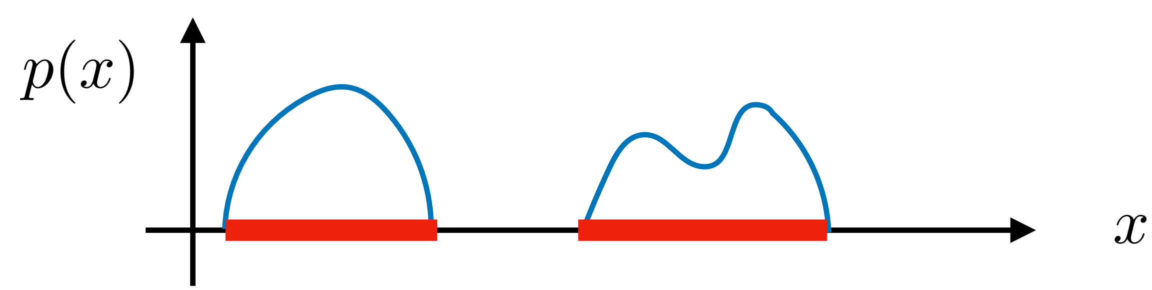


What and Where Information

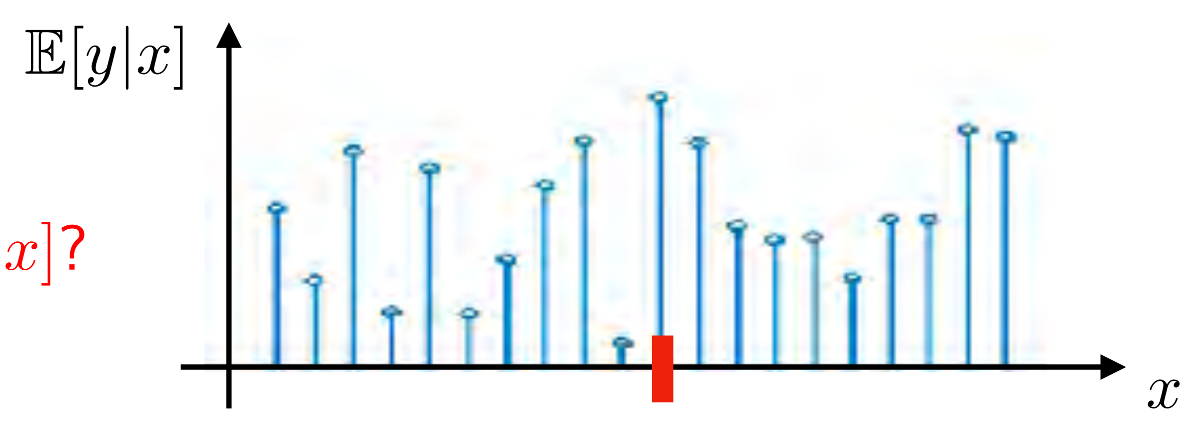
Density estimation: What is $p(y|x)$?
Classification: Where is $p(y|x) > 0$?



Density estimation: What is $p(x)$?
Clustering: Where is $p(x) > \epsilon$?



Function estimation: What is $\mathbb{E}[y|x]$?
Bandit optimization: Where is $\max_x \mathbb{E}[y|x]$?



Active learning is more efficient than passive learning for localized “where” information

Meta-Algorithm for Active Learning

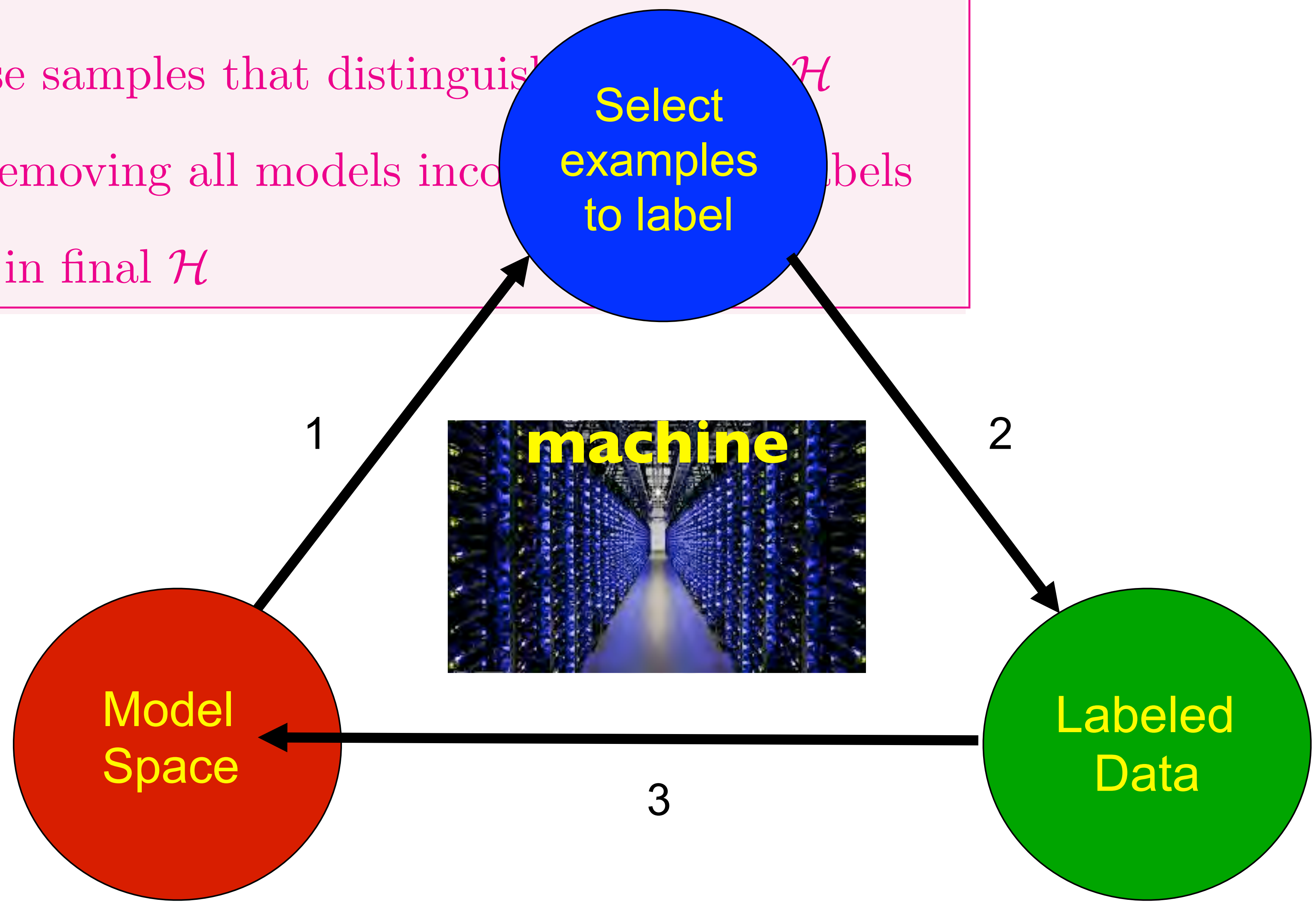
Version-Space (VS) Active Learning

initialize VS: $\mathcal{H}$ = all models/hypotheses

while (*stopping-criterion*) not met

1. **sample** at random from available dataset
2. **label** only those samples that distinguis… $\mathcal{H}$
3. **reduce** $\mathcal{H}$ by removing all models inco… …bels

output: best model in final $\mathcal{H}$



Learning a 1-D Classifier

binary search quickly finds **decision boundary**

passive : err $\sim \quad n^{-1}$

active : err $\sim \quad 2^{-n}$

Vapnik-Chervonenkis (VC) Theory

Given training data $\{(x_j, y_j)\}_{j=1}^n$, learn a function f to predict y from x

Consider a possibly infinite set of hypotheses $\mathcal{F}$ with *finite VC dimension* d and for each $f \in \mathcal{F}$ define the risk (error rate):

$$R(f) \;:=\; \mathbb{P}(f(x) \neq y)$$

error rate on training data:

$$\widehat{R}(f) \;=\; \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\Big(f(x_i) \neq y_i\Big)$$

"empirical risk"

VC bound:

$$\sup_{f \in \mathcal{F}} |R(f) - \widehat{R}(f)| \;\leq\; 6\sqrt{\frac{d \log(n/\delta)}{n}}$$

w.p. $\geq\; 1 - \delta$

Empirical Risk Minimization (ERM)

Goal: select hypothesis with true error rate within $\epsilon > 0$ of $\min_{f \in \mathcal{F}} R(f)$

$$f^* = \arg\min_{f \in \mathcal{F}} R(f)$$ true risk minimizer

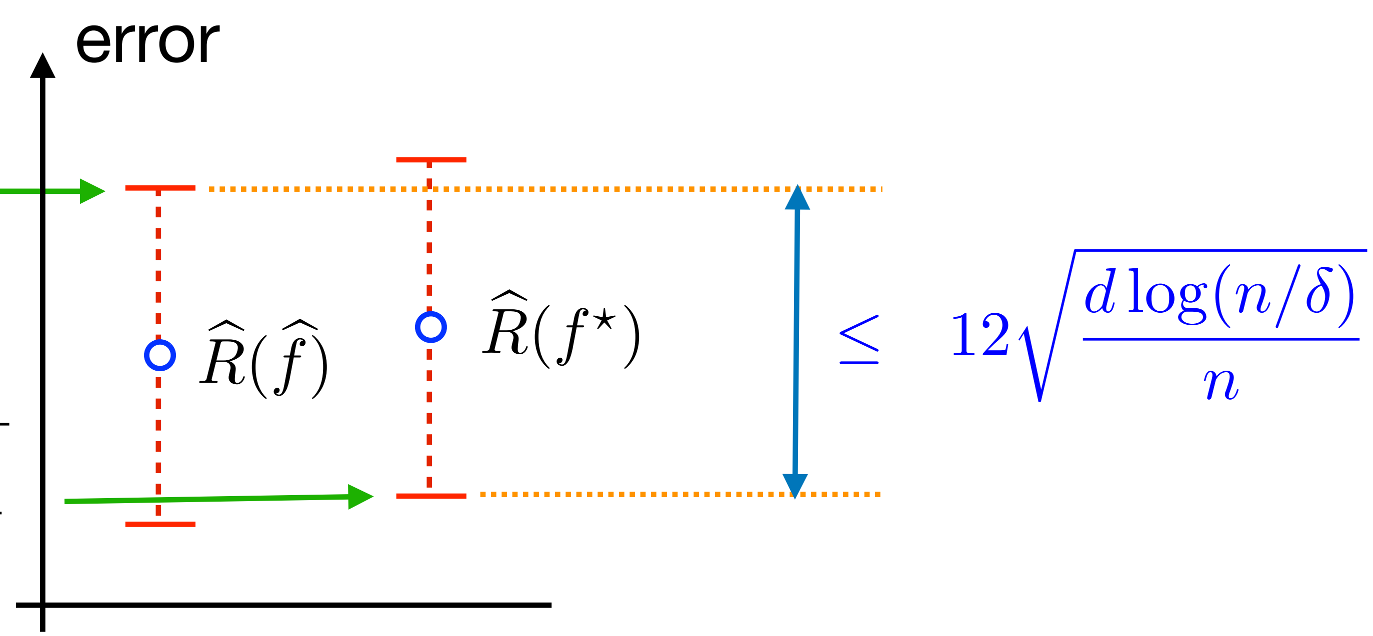
$$\widehat{f} = \arg\min_{f \in \mathcal{F}} \widehat{R}(f)$$ empirical risk minimizer

$\widehat{f}$ minimizes empirical risk:

$$\widehat{R}(\widehat{f}) \leq \widehat{R}(f^*)$$

$$R(\widehat{f}) \leq \widehat{R}(\widehat{f}) + 6\sqrt{\frac{d \log(n/\delta)}{n}}$$

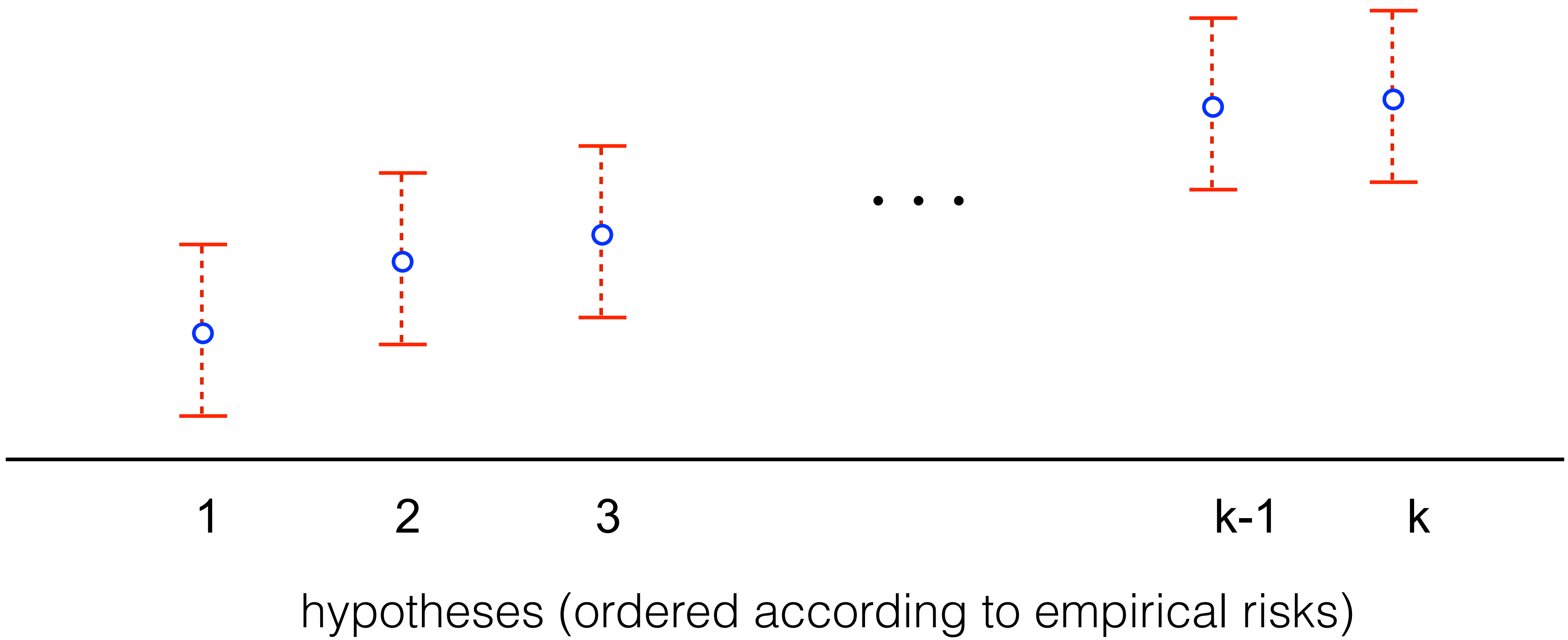
$$R(f^*) \geq \widehat{R}(f^*) - 6\sqrt{\frac{d \log(n/\delta)}{n}}$$



sufficient number of training examples:

$$12\sqrt{\frac{d \log(n/\delta)}{n}} \leq \epsilon \quad \Rightarrow \quad n = \widetilde{O}\left(\frac{d \log(1/\delta)}{\epsilon^2}\right)$$

Empirical Risks and Confidence Intervals



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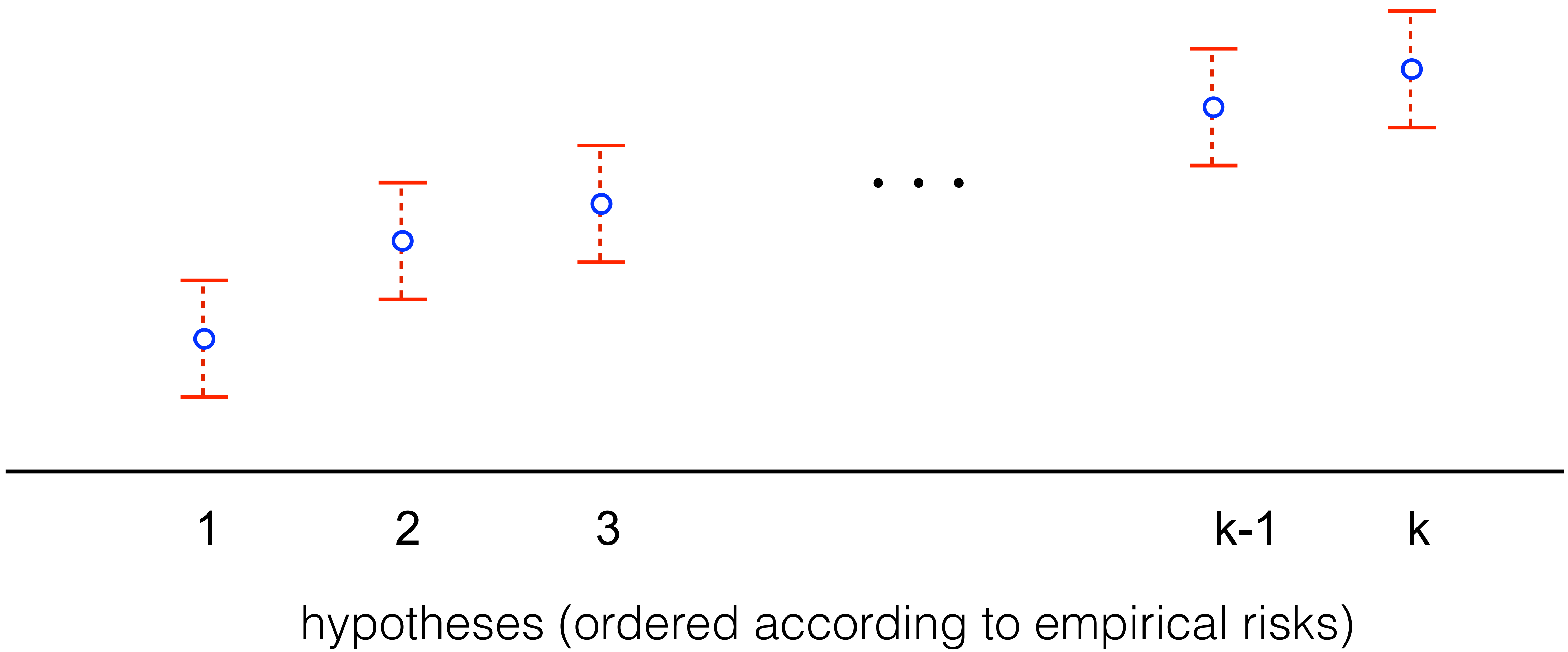
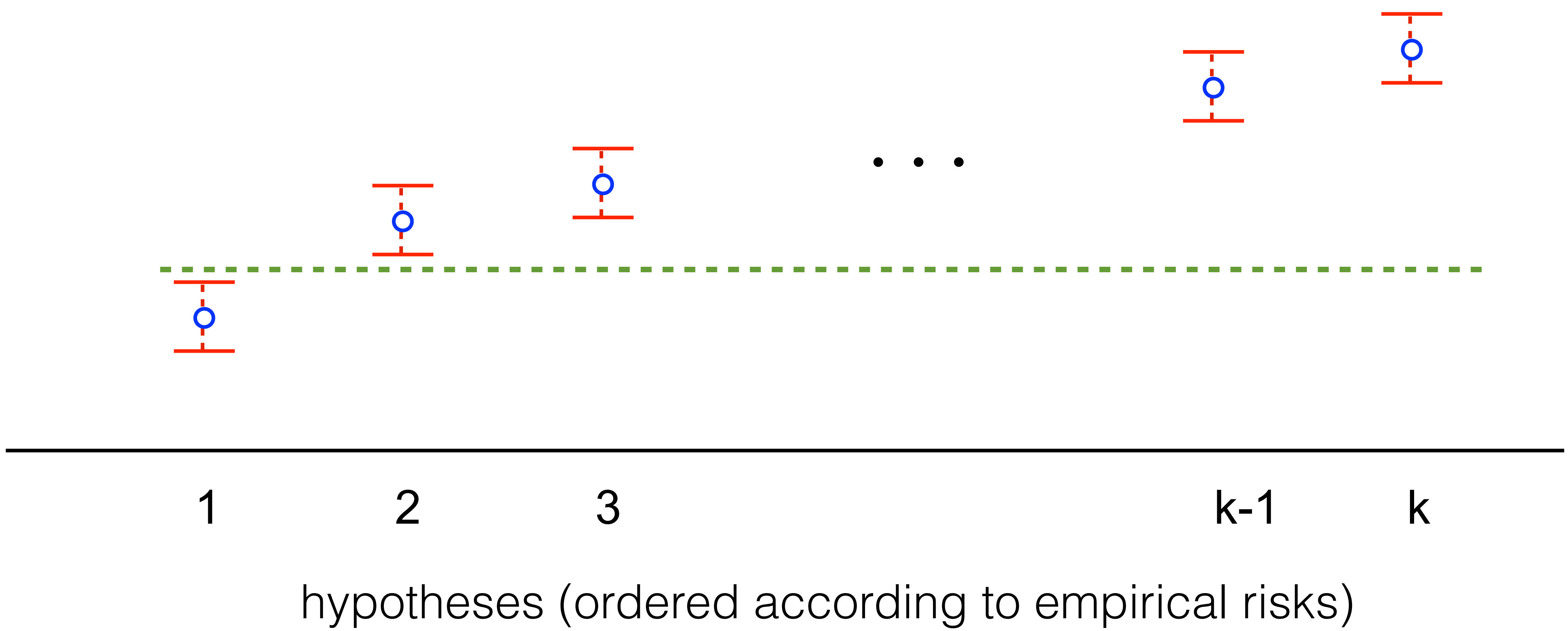


more training data $\Rightarrow$ smaller confidence intervals

Empirical Risks and Confidence Intervals



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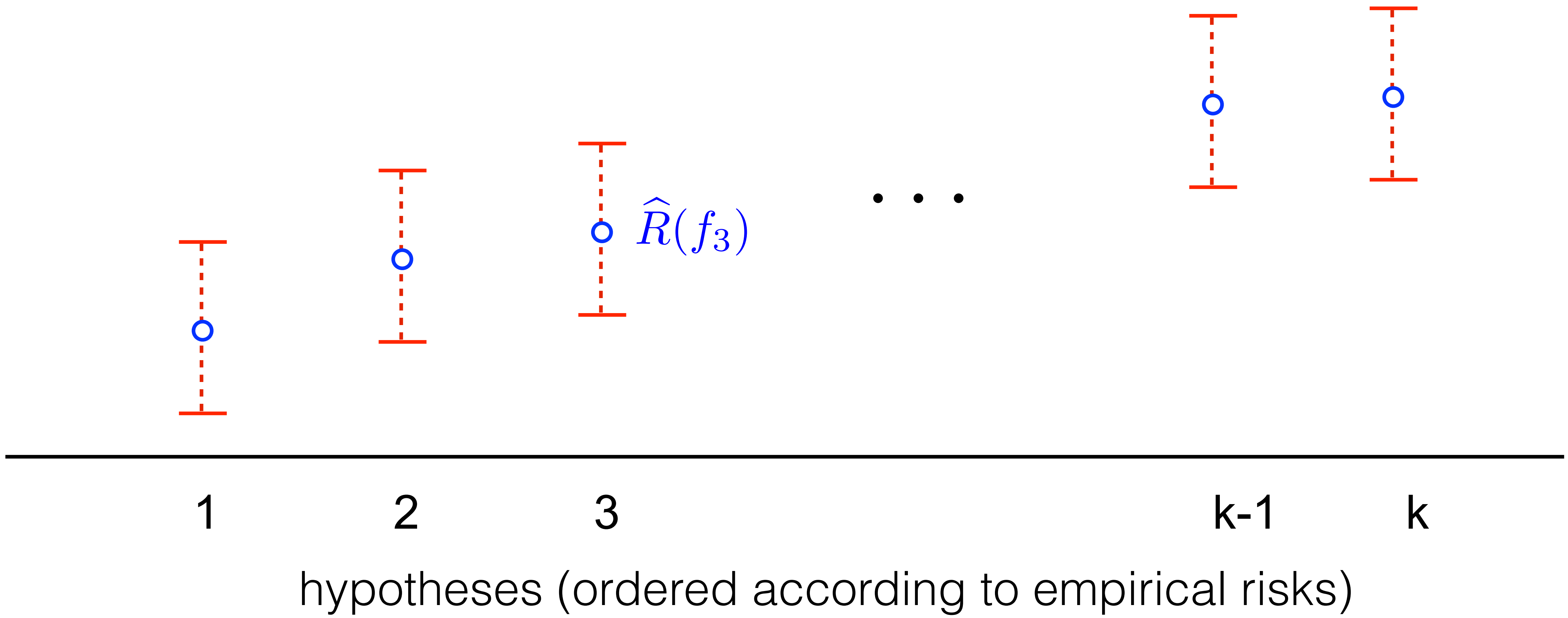
ERM is Wasting Labeled Examples



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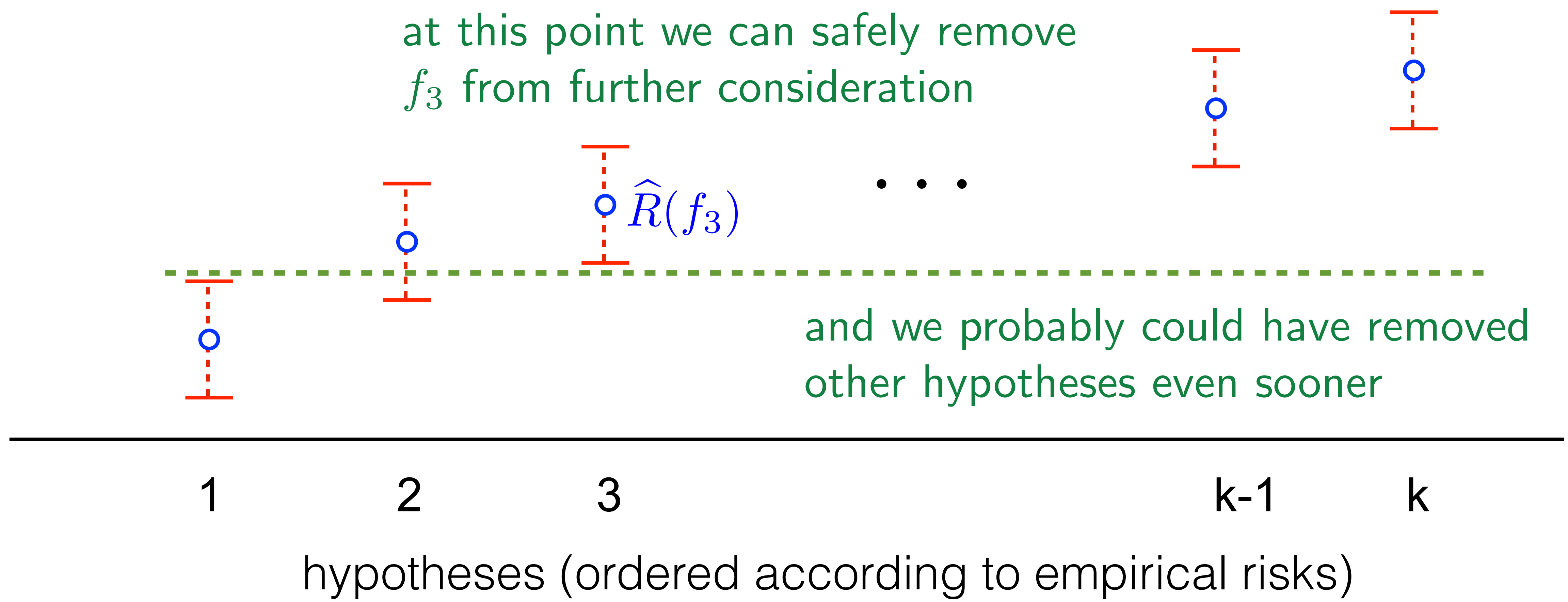
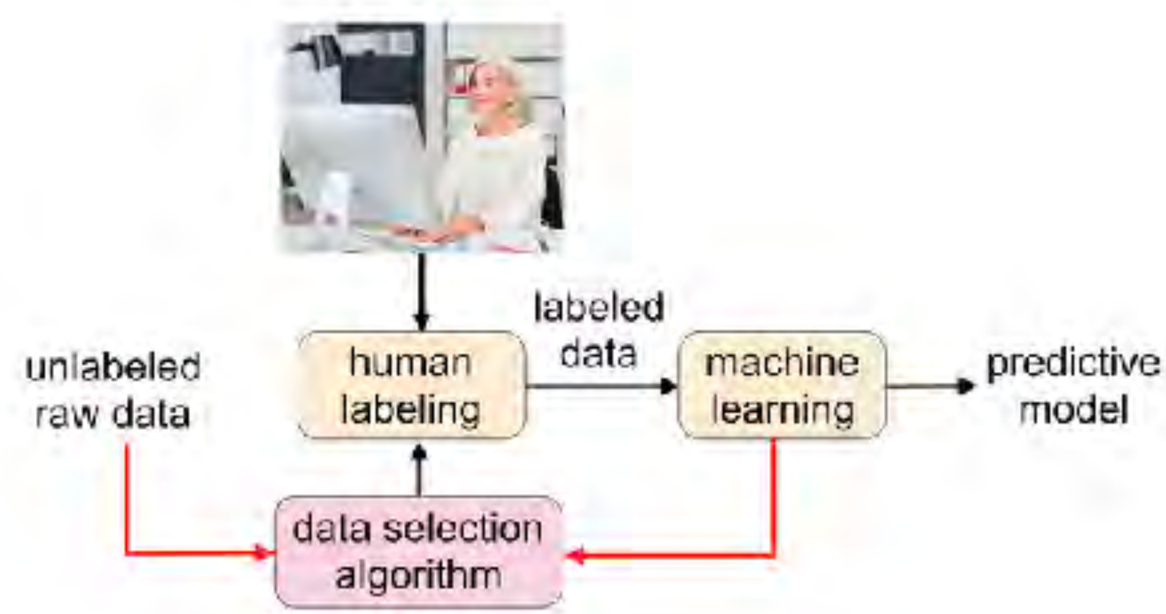
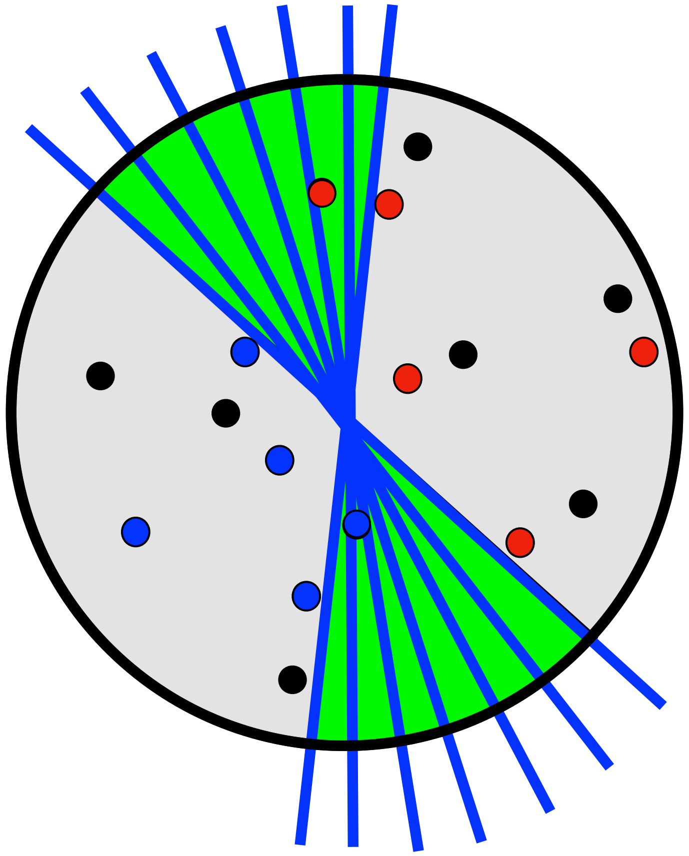


only require labels for examples that hypotheses 1 and 2 label differently (i.e., examples where they *disagree*)



Disagreement-Based Active Learning

consider points uniform on unit ball and
linear classifiers passing through origin



only label points in the
region of disagreement $\mathfrak{D}$

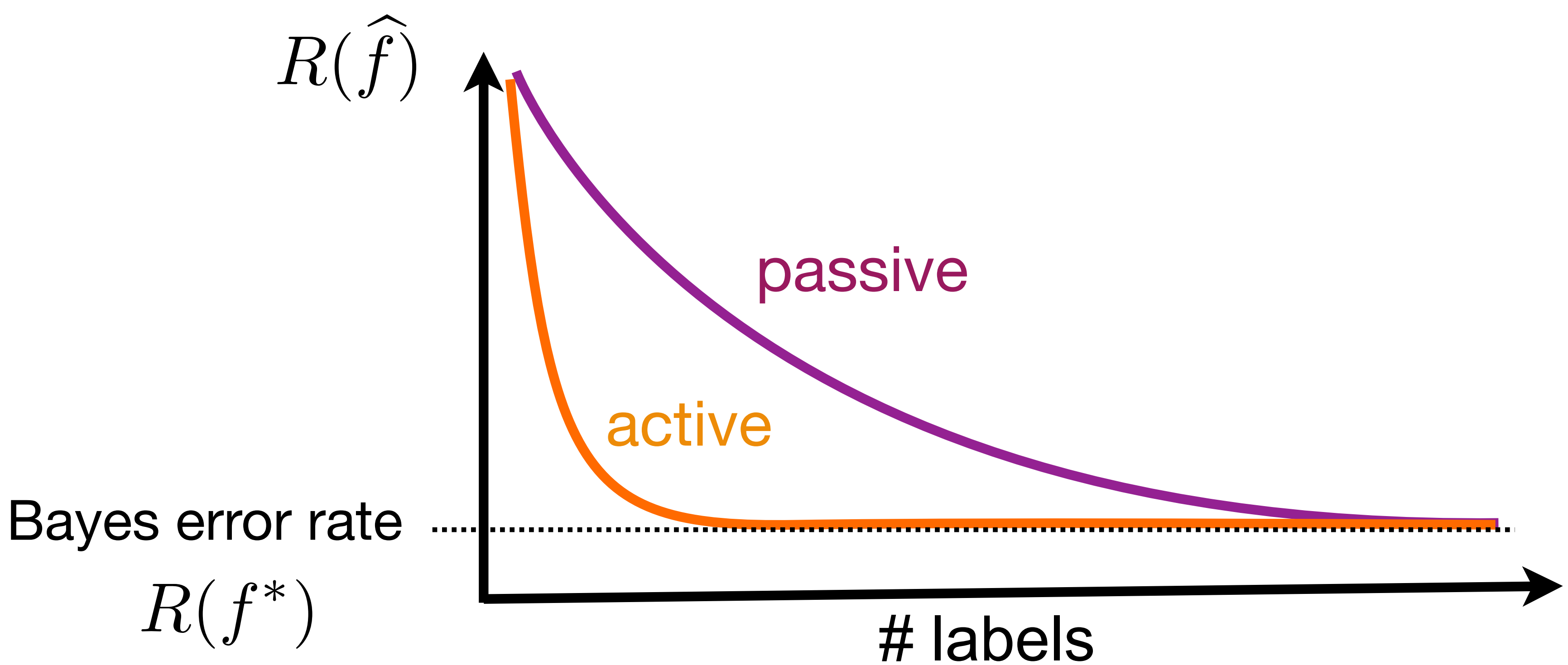
Active Binary Classification

Assuming optimal Bayes classifer f^* in VC class with dimension d and "nice" distributions (e.g., bounded label noise)

$$\epsilon \quad = \quad R(\widehat{f}) - R(f^*)$$

passive $\quad \epsilon \sim \frac{d}{n} \quad$ parametric rate

active $\quad \epsilon \sim \exp\left(-c\,\frac{n}{d}\right) \quad$ exponential speed-up



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Recommended Reading (Foundations of Active Learning)

Settles, Burr. "Active learning." *Synthesis Lectures on Artificial Intelligence and Machine Learning* 6.1 (2012): 1-114.

Dasgupta, Sanjoy. "Two faces of active learning." *Theoretical computer science* 412.19 (2011): 1767-1781.

Cohn, David, Les Atlas, and Richard Ladner. "Improving generalization with active learning." *Machine learning* 15.2 (1994): 201-221.

Castro, Rui M., and Robert D. Nowak. "Minimax bounds for active learning." *IEEE Transactions on Information Theory* 54, no. 5 (2008): 2339-2353.

Zhu, Xiaojin, John Lafferty, and Zoubin Ghahramani. "Combining active learning and semi-supervised learning using gaussian fields and harmonic functions." *ICML 2003 workshop*. Vol. 3. 2003.

Dasgupta, Sanjoy, Daniel J. Hsu, and Claire Monteleoni. "A general agnostic active learning algorithm." *Advances in neural information processing systems*. 2008.

Balcan, Maria-Florina, Alina Beygelzimer, and John Langford. "Agnostic active learning." *Journal of Computer and System Sciences* 75.1 (2009): 78-89.

Nowak, Robert D. "The geometry of generalized binary search." *IEEE Transactions on Information Theory* 57, no. 12 (2011): 7893-7906.

Hanneke, Steve. "Theory of active learning." *Foundations and Trends in Machine Learning* 7, no. 2-3 (2014).

Part 2: Theory of Active Learning
General Case

- Disagreement-Based Agnostic Active Learning
- Disagreement Coefficient
- Sample Complexity Bounds

Tutorial on Active Learning: Theory to Practice

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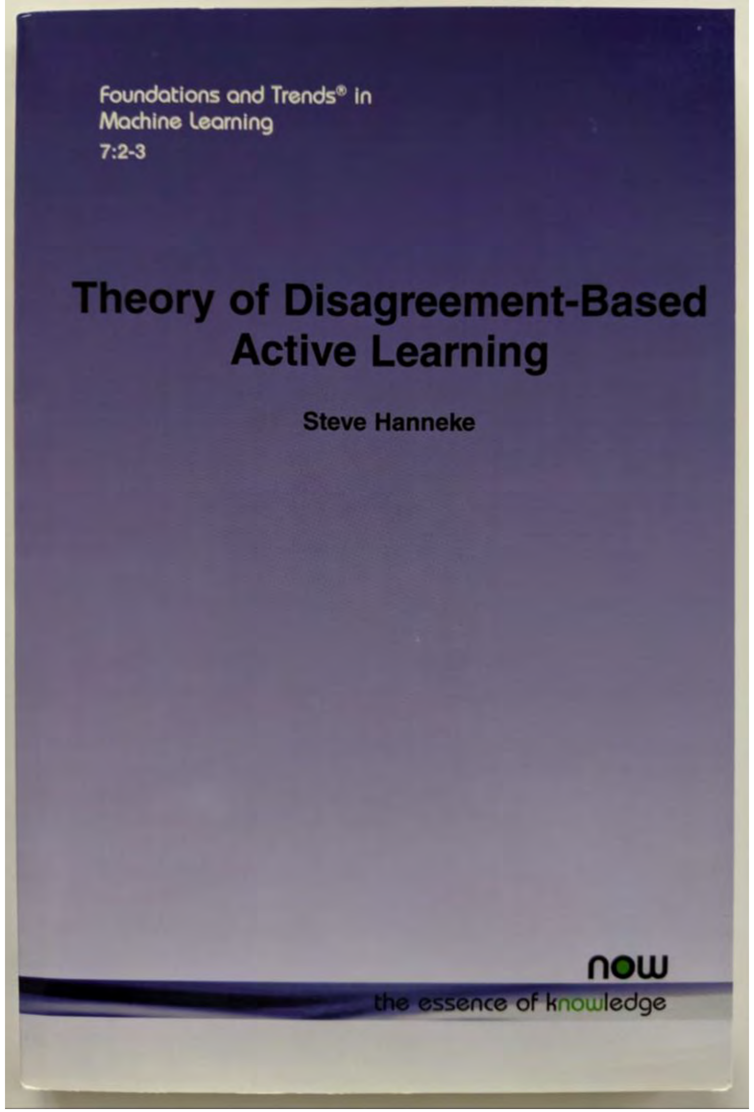
Agnostic Active Learning



Uniform Bernstein Inequality

Bernstein's inequality:

For m iid samples
$\forall f, f'$, w.p. $1-\delta$,
$$R(f) - R(f') \leq \hat{R}(f) - \hat{R}(f') + c\sqrt{\hat{P}(f \neq f')\frac{\log(1/\delta)}{m}} + \frac{\log(1/\delta)}{m}$$

Uniform Bernstein inequality:

VC dimension

w.p. $1-\delta$, $\forall f, f' \in \mathcal{H}$,
$$R(f) - R(f') \leq \hat{R}(f) - \hat{R}(f') + c\sqrt{\hat{P}(f \neq f')\frac{d\log(m/\delta)}{m}} + \frac{d\log(m/\delta)}{m}$$

Roughly:
$\forall f, f' \in \mathcal{H}$,
$$R(f) - R(f') \leq \hat{R}(f) - \hat{R}(f') + \sqrt{\hat{P}(f \neq f')\frac{d}{m}}$$

Agnostic Active Learning

Balcan, Beygelzimer, & Langford (2006)

Region of disagreement:

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
2. **label** points in $Q = \mathrm{DIS}(\mathcal{H}) \cap S$
3. **optimize** $\hat{f} \leftarrow \underset{f \in \mathcal{H}}{\mathrm{argmin}}\, \hat{R}_Q(f)$
4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$.

output final $\hat{f}$

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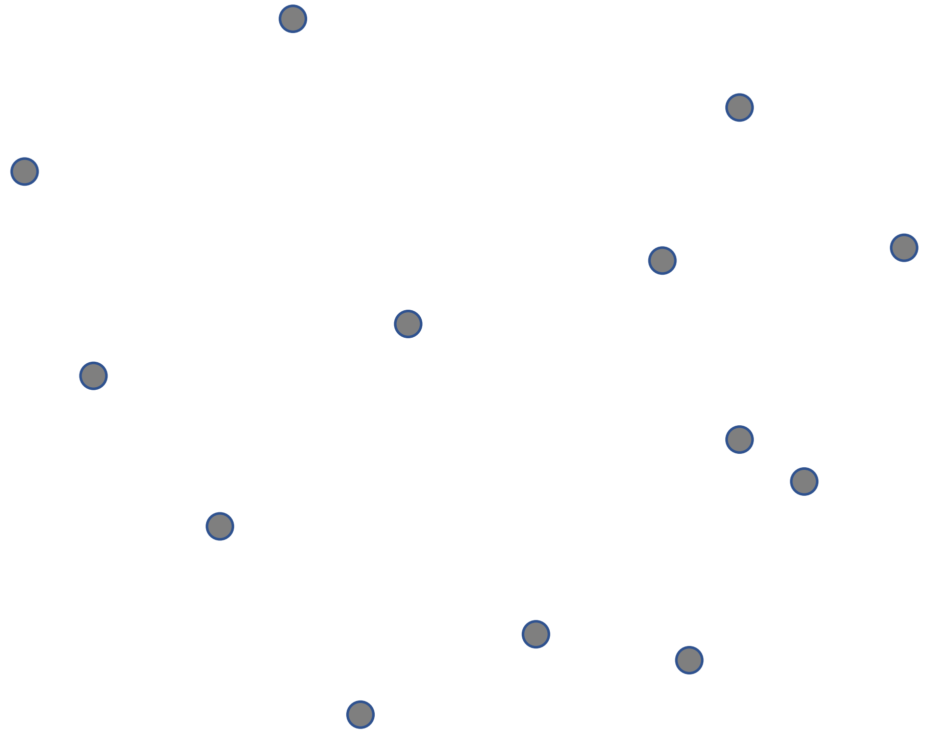

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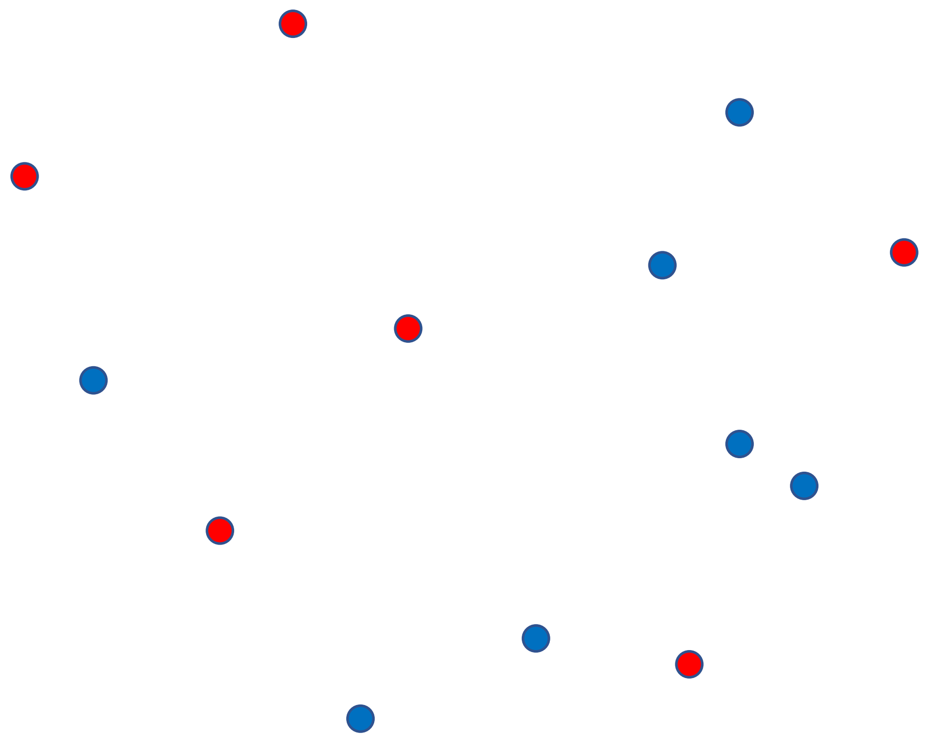

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***A*² (Agnostic Active)**

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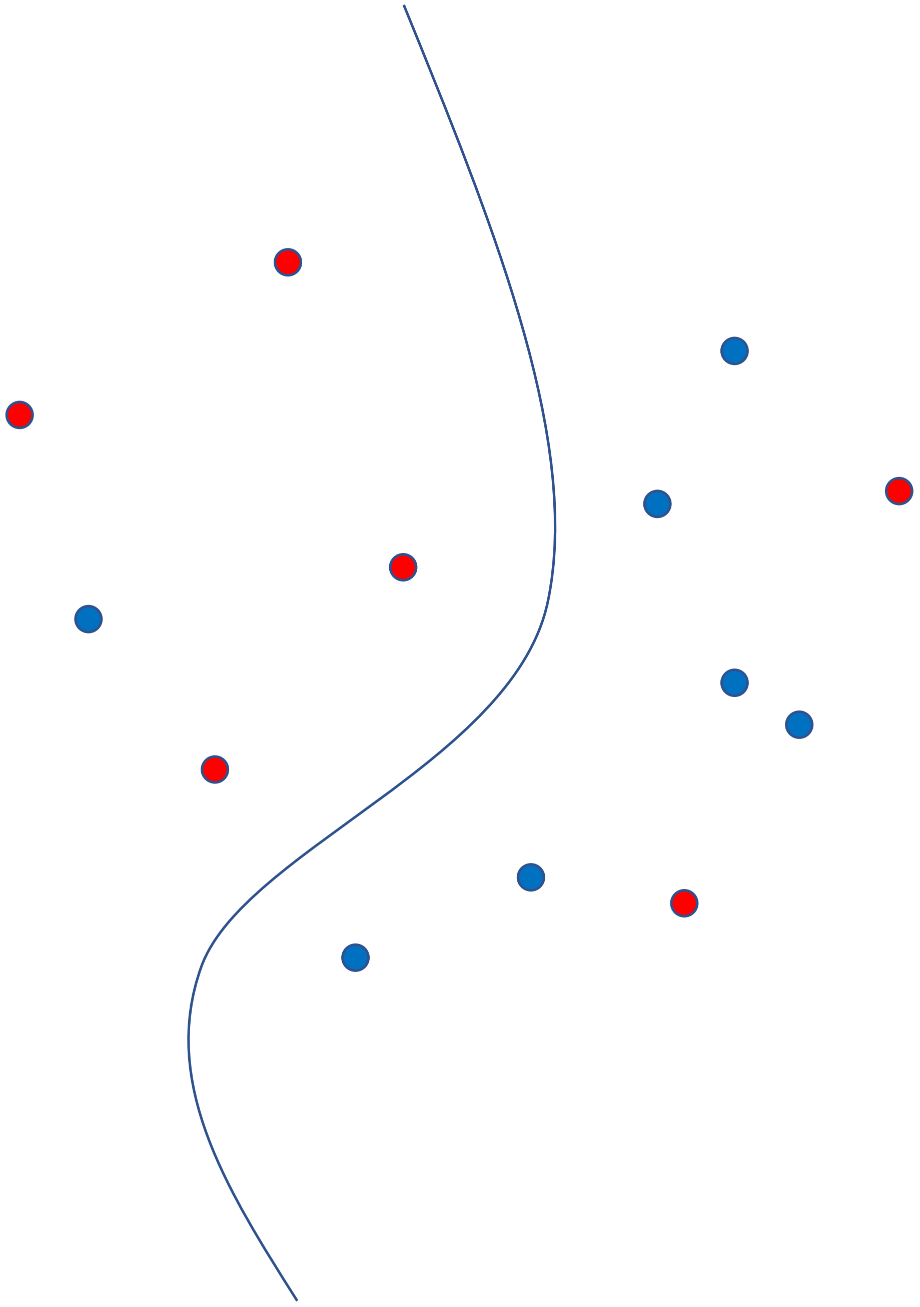

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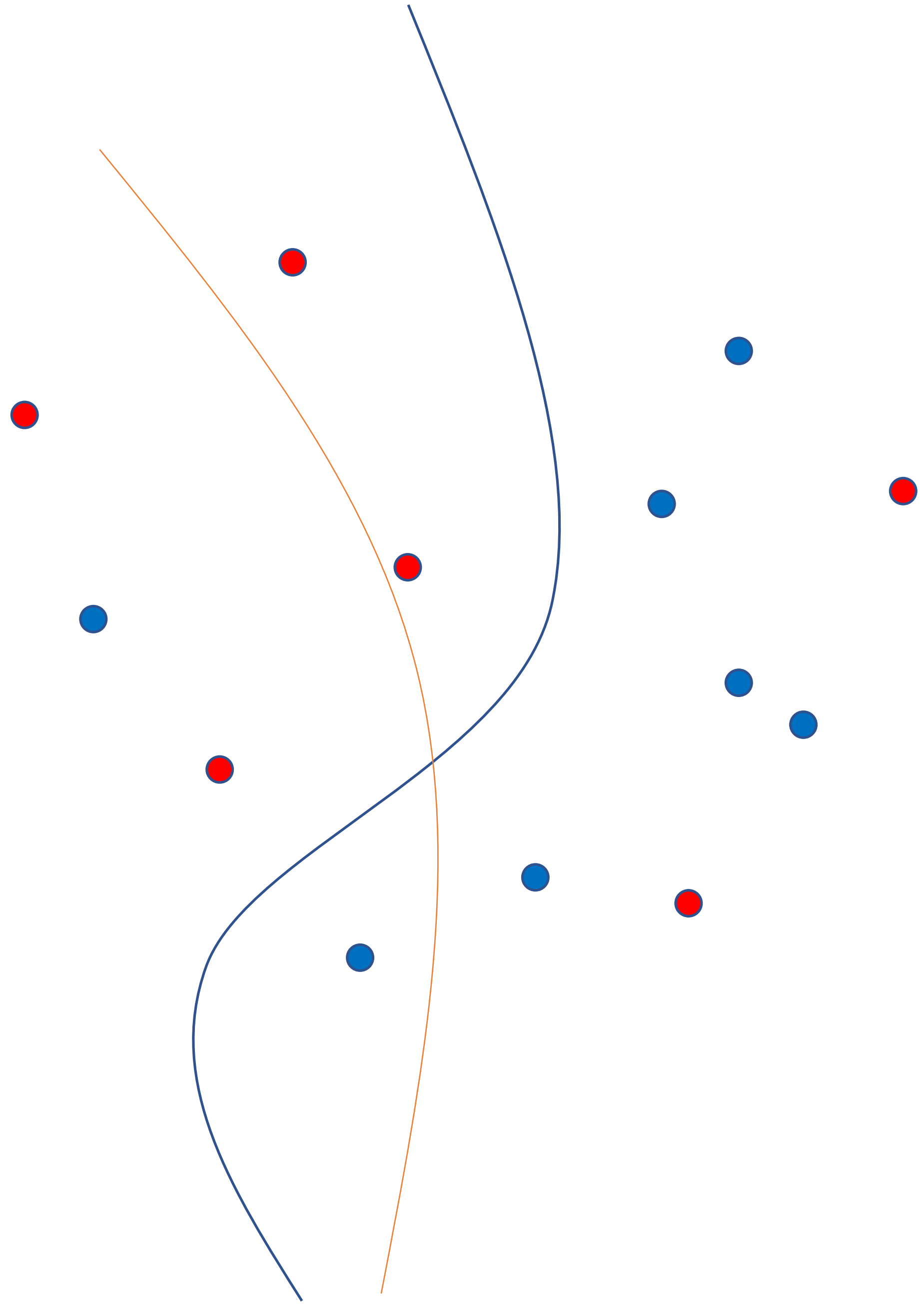

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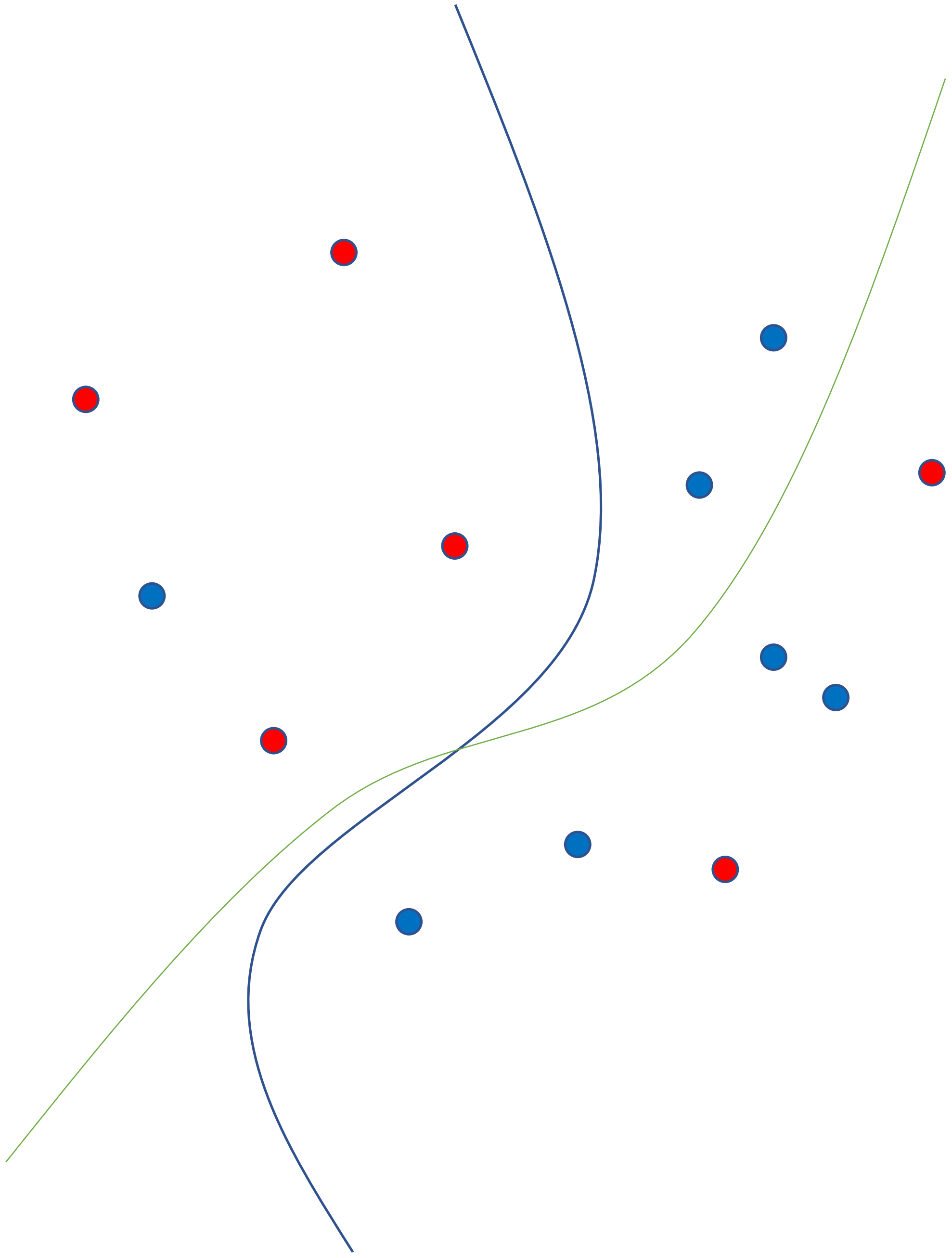

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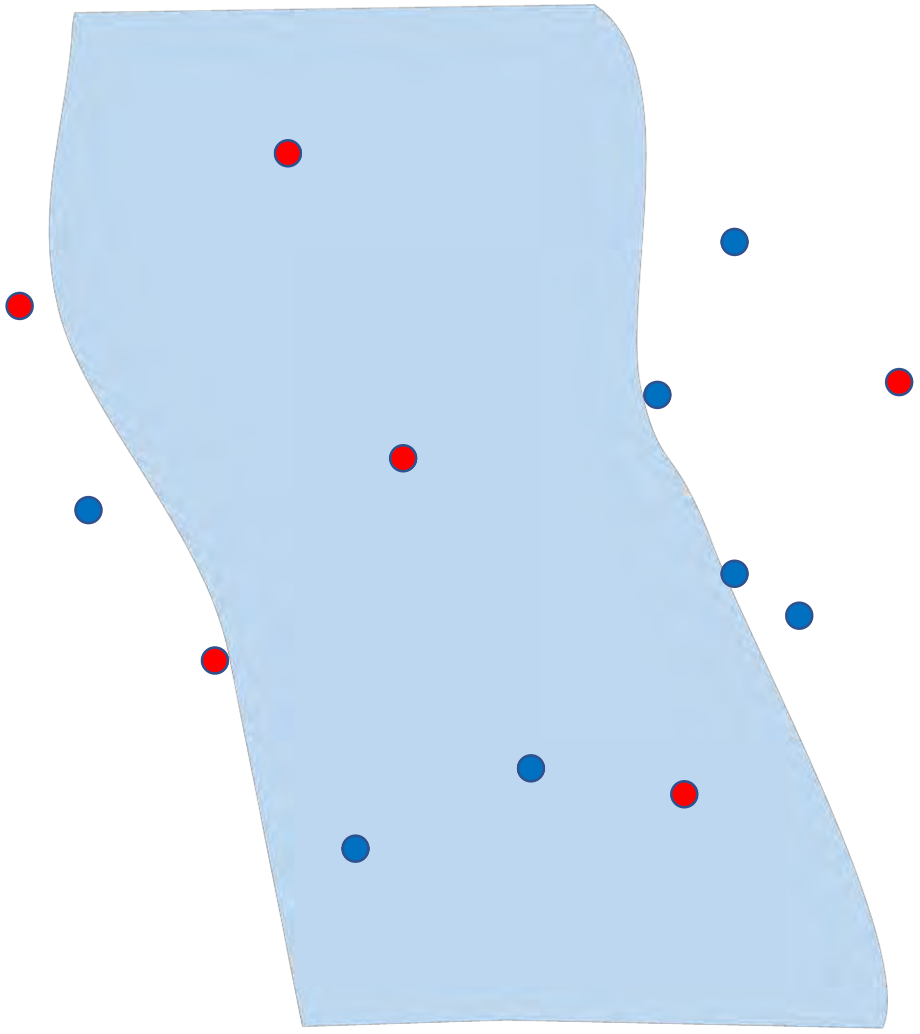

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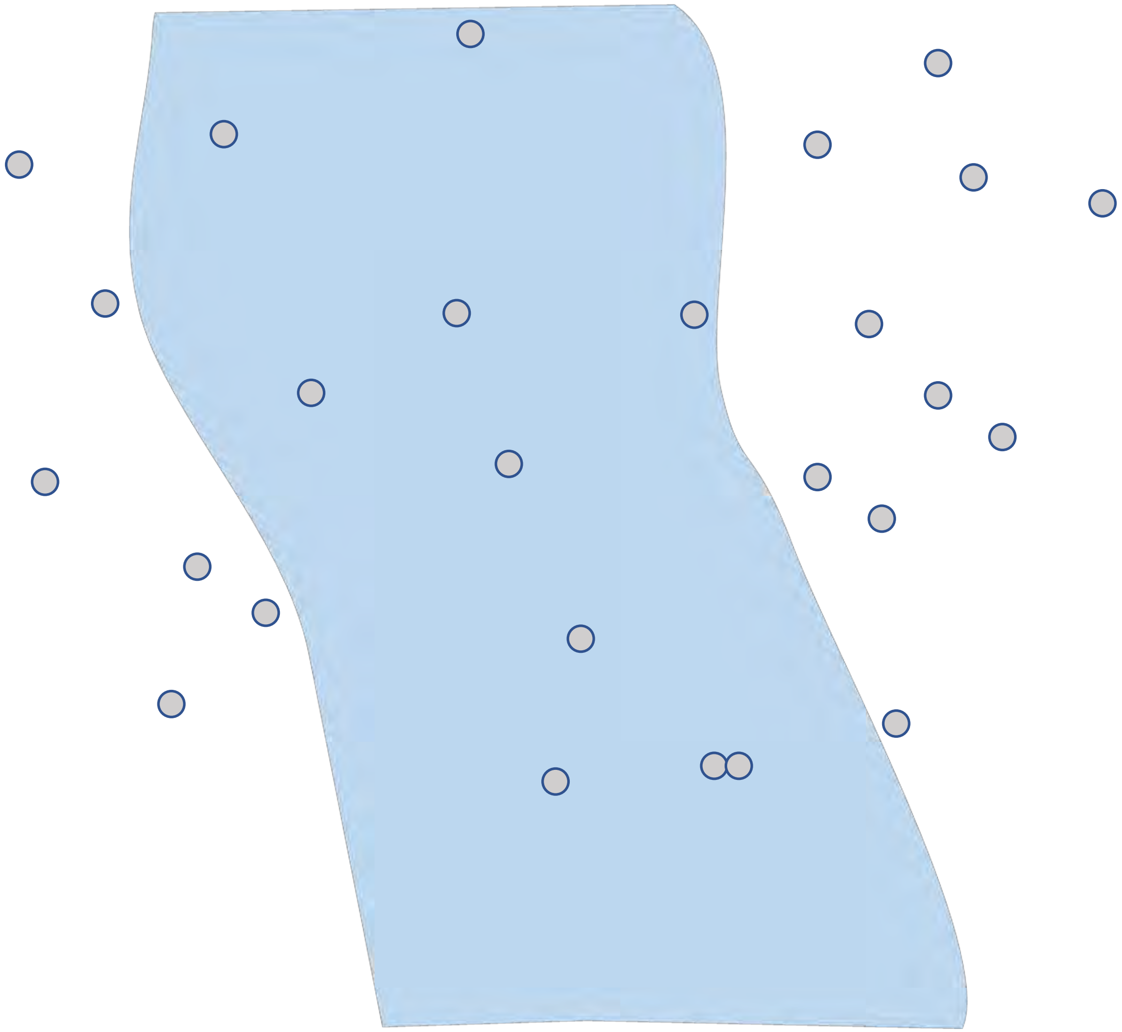

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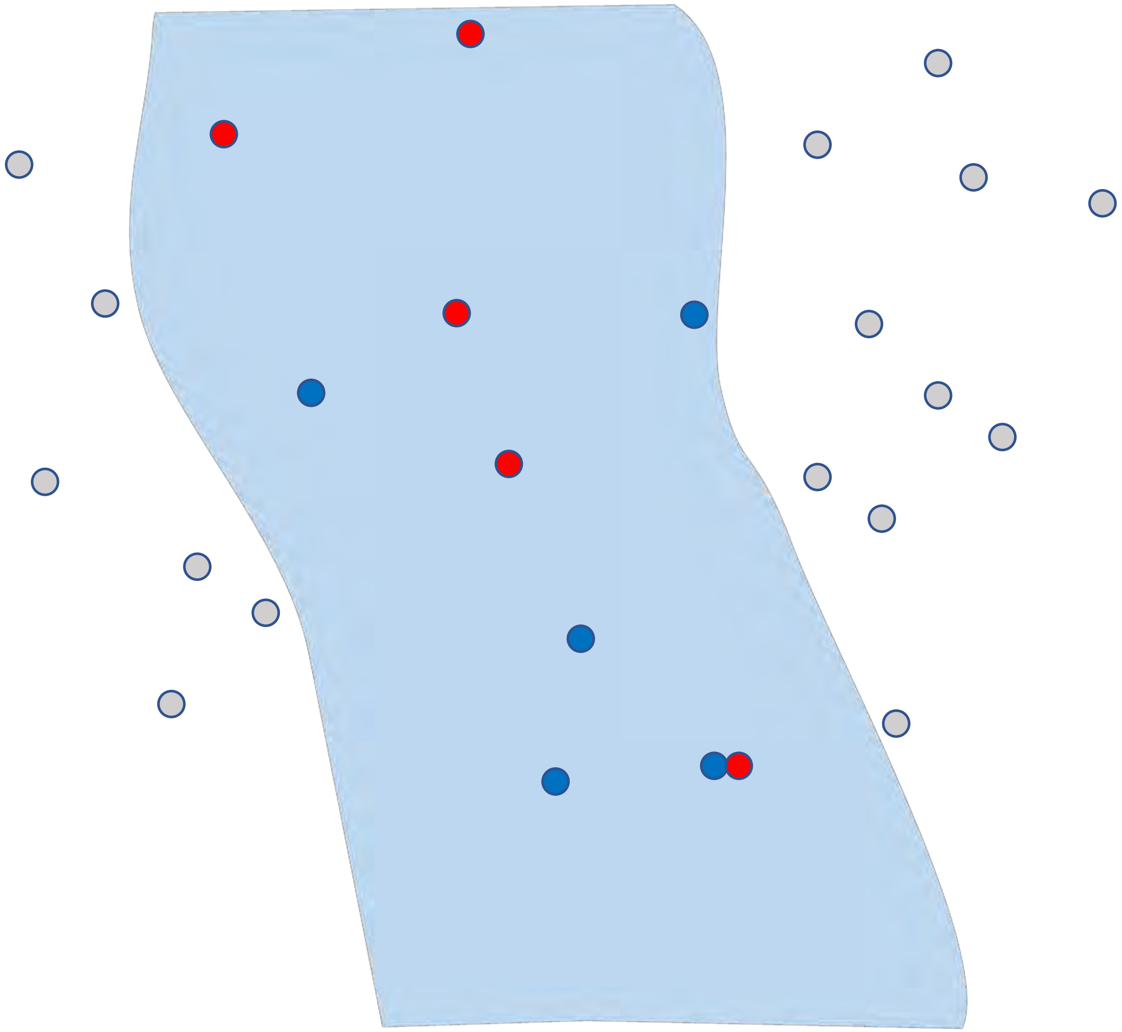

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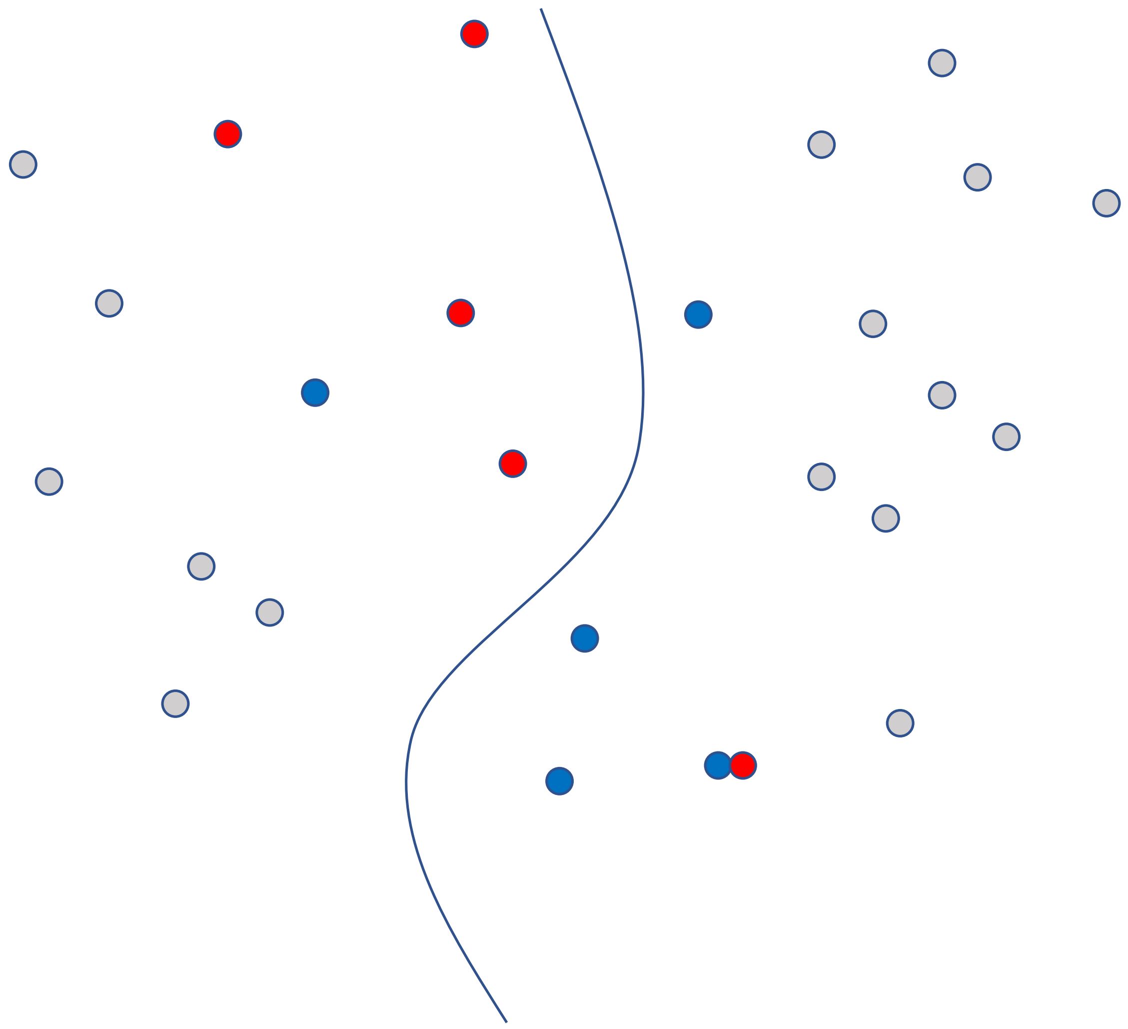
output final $\hat{f}$



Agnostic Active Learning

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
2. **label** points in $Q = \mathrm{DIS}(\mathcal{H}) \cap S$
3. **optimize** $\hat{f} \leftarrow \underset{f \in \mathcal{H}}{\mathrm{argmin}}\, \hat{R}_Q(f)$
4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$.

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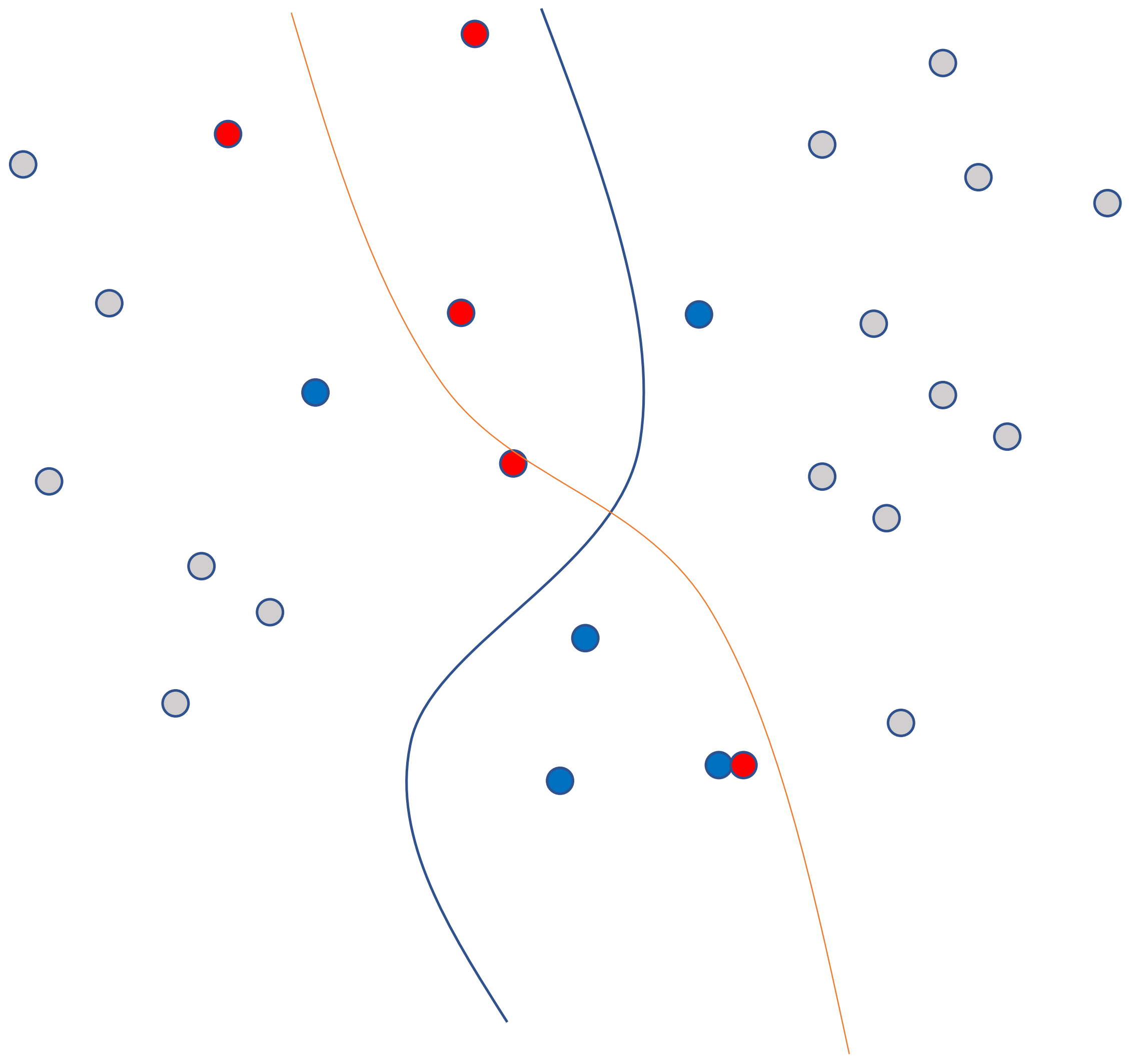

Agnostic Active Learning

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

***A*²** **(Agnostic Active)**

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
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3. **optimize** $\hat{f} \leftarrow \underset{f \in \mathcal{H}}{\operatorname{argmin}} \hat{R}_Q(f)$
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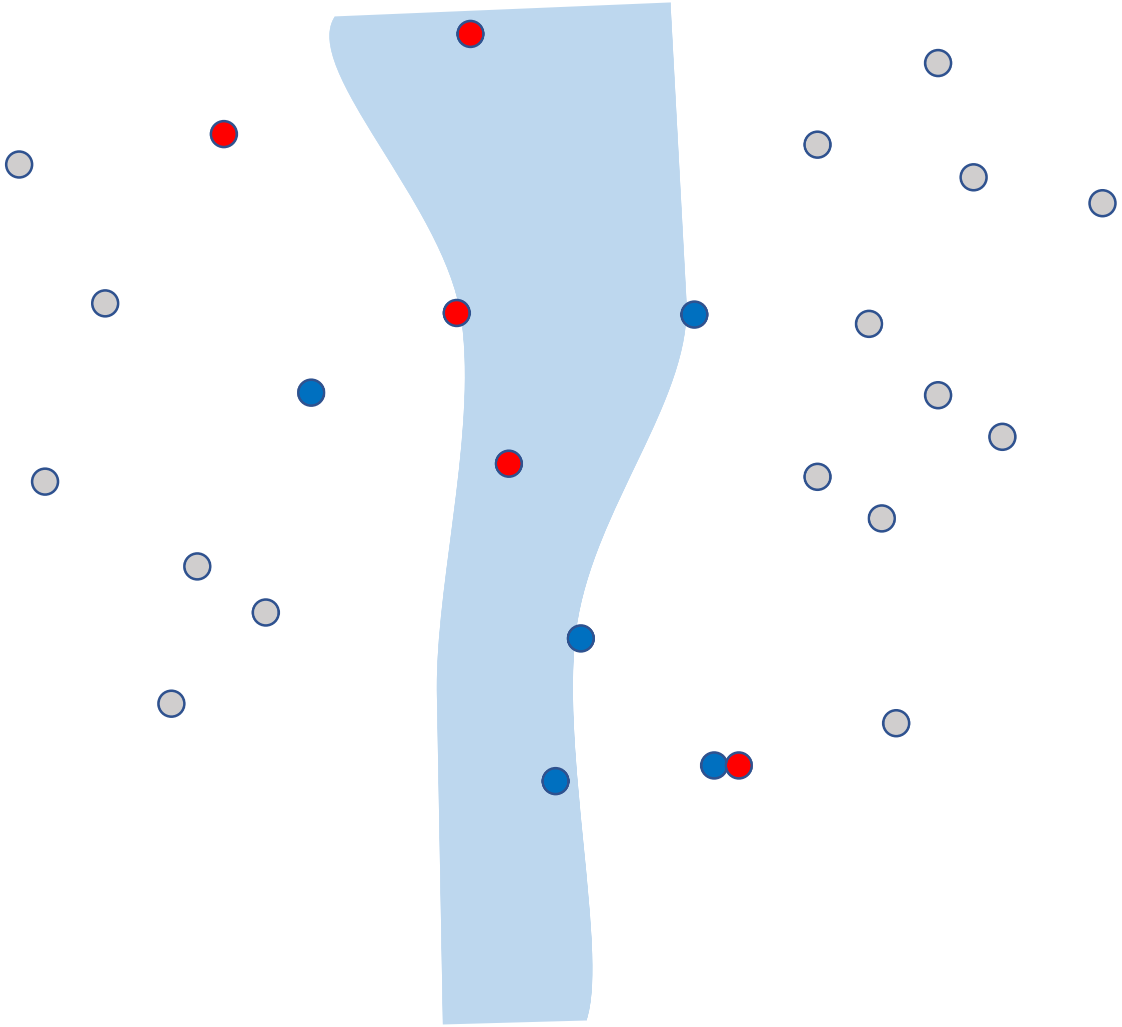
output final $\hat{f}$



Agnostic Active Learning

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A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

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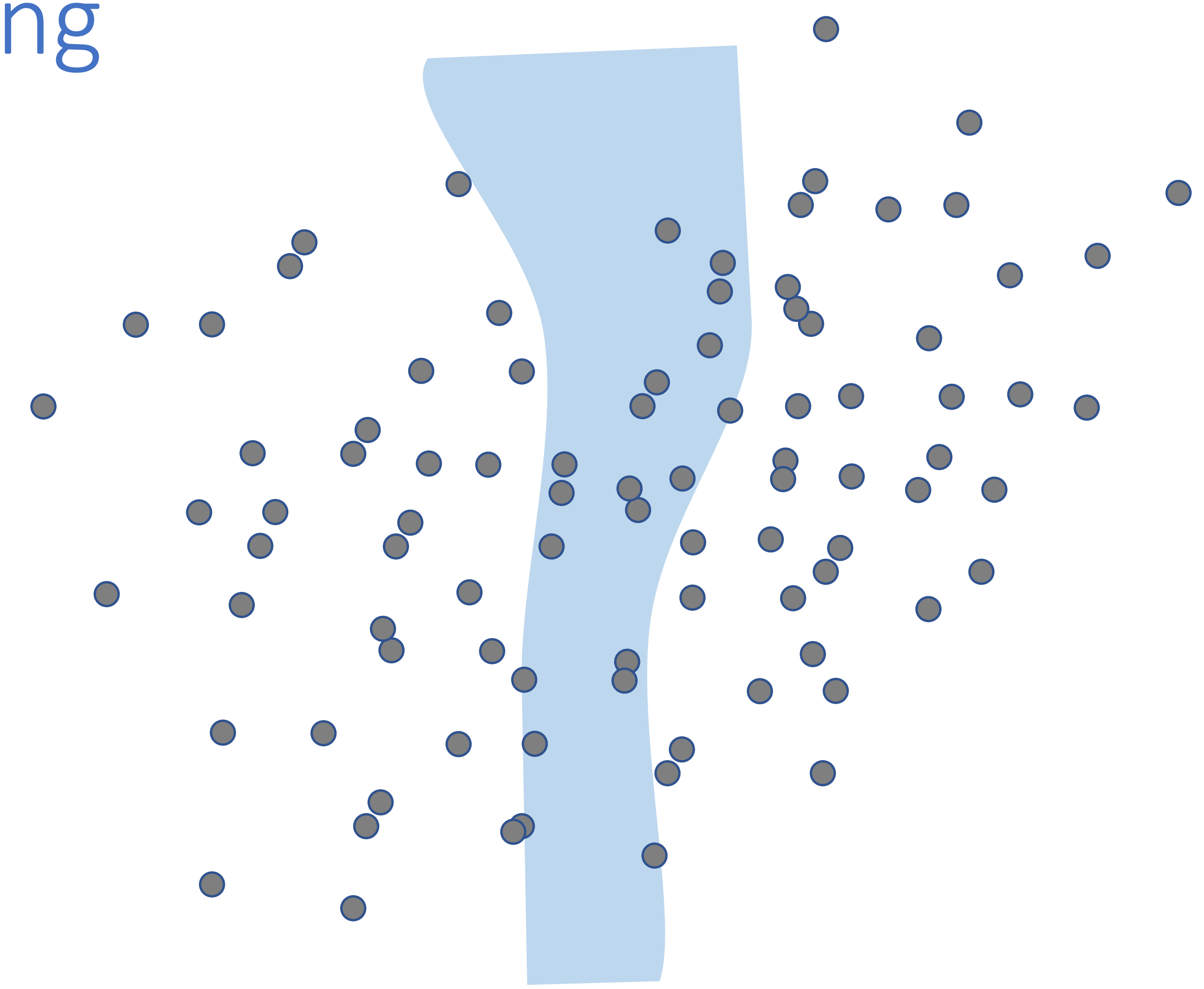
output final $\hat{f}$



Agnostic Active Learning

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
2. **label** points in $Q = \mathrm{DIS}(\mathcal{H}) \cap S$
3. **optimize** $\hat{f} \leftarrow \operatorname*{argmin}_{f \in \mathcal{H}} \hat{R}_Q(f)$
4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$

output final $\hat{f}$

The point:

Any t with $f^* \in \mathcal{H}$ still,

$R(f^*|\mathrm{DIS}(\mathcal{H}))$ still **minimal** in $\mathcal{H}$

$\Rightarrow$

$$\hat{R}_Q(f^*) - \hat{R}_Q(\hat{f})$$
$$\leq R(f^*|\mathrm{DIS}(\mathcal{H})) - R(\hat{f}|\mathrm{DIS}(\mathcal{H})) + \sqrt{\hat{P}_Q(f^* \neq \hat{f}) \frac{d}{|Q|}}$$
$$\leq \sqrt{\hat{P}_Q(f^* \neq \hat{f}) \frac{d}{|Q|}}$$

$\Rightarrow$ **f^* never removed.**

Agnostic Active Learning

$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
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The point:
Any t with $f^* \in \mathcal{H}$ still,
$R(f^*|\mathrm{DIS}(\mathcal{H}))$ still **minimal** in $\mathcal{H}$

$\Rightarrow$
$$\hat{R}_Q(f^*) - \hat{R}_Q(\hat{f})$$
$$\leq R(f^*|\mathrm{DIS}(\mathcal{H})) - R(\hat{f}|\mathrm{DIS}(\mathcal{H})) + \sqrt{\hat{P}_Q(f^* \neq \hat{f}) \frac{d}{|Q|}}$$
$$\leq \sqrt{\hat{P}_Q(f^* \neq \hat{f}) \frac{d}{|Q|}}$$

$\Rightarrow$ **f^* never removed.**

Next: **How many labels does it use?**

Sample Complexity Analysis

Hanneke (2007,…)

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

$\mathrm{DIS}(\mathrm{B}(f^*, r)) := \{x \in \mathcal{X} : \exists f, f' \in \mathrm{B}(f^*, r), f(x) \neq f'(x)\}$

Disagreement coefficient:

$$\theta = \sup_{r > \epsilon} \frac{P_X(\mathrm{DIS}(\mathrm{B}(f^*, r)))}{r}$$

Sample Complexity Analysis

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

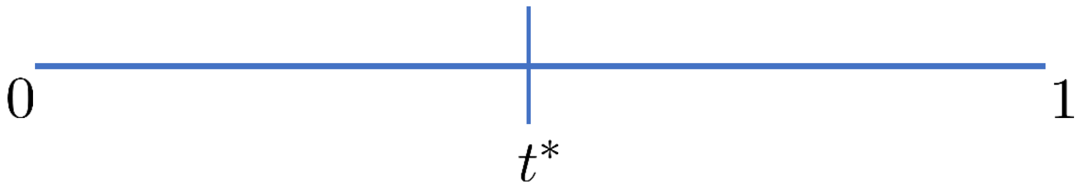
$\mathrm{DIS}(\mathrm{B}(f^*, r)) := \{x \in \mathcal{X} : \exists f, f' \in \mathrm{B}(f^*, r), f(x) \neq f'(x)\}$

Disagreement coefficient:

$$\theta = \sup_{r > \epsilon} \frac{P_X(\mathrm{DIS}(\mathrm{B}(f^*, r)))}{r}$$

Example: **Thresholds**, P_X Uniform$(0, 1)$

$f(x) = \mathbb{I}[x \geq t]$



Sample Complexity Analysis

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

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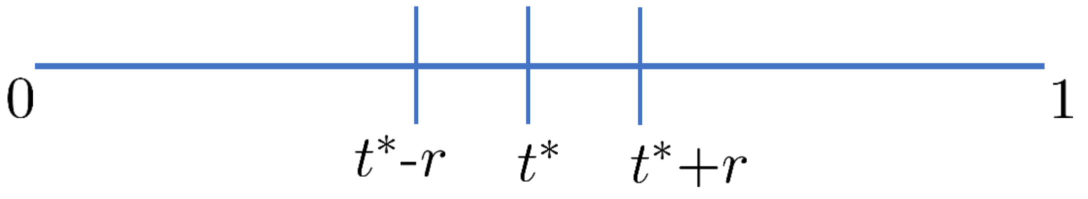

$$\mathrm{DIS}(\mathrm{B}(f^*, r)) = [t^* - r, t^* + r)$$

$$P_X(\mathrm{DIS}(\mathrm{B}(f^*, r))) = 2r$$

$$\theta = 2$$

Sample Complexity Analysis

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

$\mathrm{DIS}(\mathrm{B}(f^*, r)) := \{x \in \mathcal{X} : \exists f, f' \in \mathrm{B}(f^*, r), f(x) \neq f'(x)\}$

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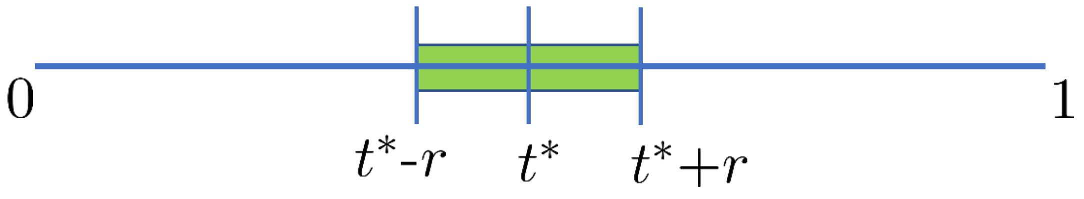
$$\theta = \sup_{r > \epsilon} \frac{P_X(\mathrm{DIS}(\mathrm{B}(f^*, r)))}{r}$$

Example: **Thresholds**, P_X Uniform$(0,1)$

$f(x) = \mathbb{I}[x \geq t]$



$$\mathrm{DIS}(\mathrm{B}(f^*, r)) = [t^* - r, t^* + r)$$

$$P_X(\mathrm{DIS}(\mathrm{B}(f^*, r))) = 2r$$

$$\Rightarrow \theta = 2$$

Sample Complexity Analysis

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

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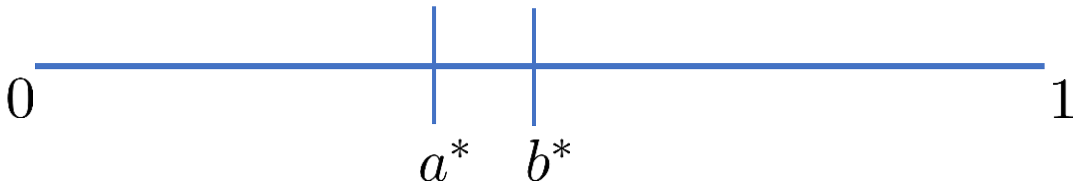
Disagreement coefficient:

$$\theta = \sup_{r > \epsilon} \frac{P_X(\mathrm{DIS}(\mathrm{B}(f^*, r)))}{r}$$

Example: **Intervals**, P_X Uniform$(0, 1)$

$f(x) = \mathbb{I}[a \leq x \leq b]$



Sample Complexity Analysis

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

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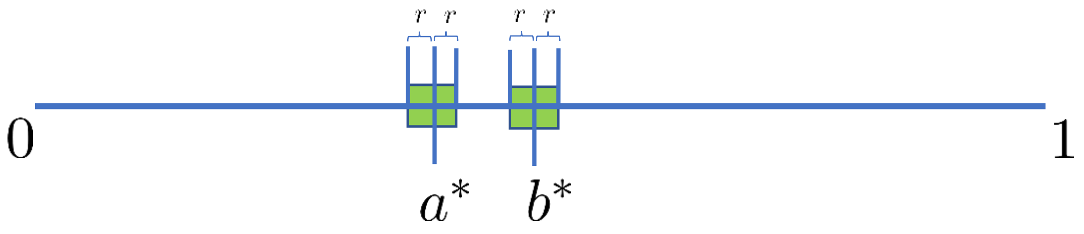


$w^* := b^* - a^*$

If $r < w^*$,

$\mathrm{DIS}(\mathrm{B}(f^*, r)) = [a^* - r, a^* + r) \cup (b^* - r, b^* + r]$

$P_X(\mathrm{DIS}(\mathrm{B}(f^*, r))) = 4r$

Sample Complexity Analysis

Ball: $B(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

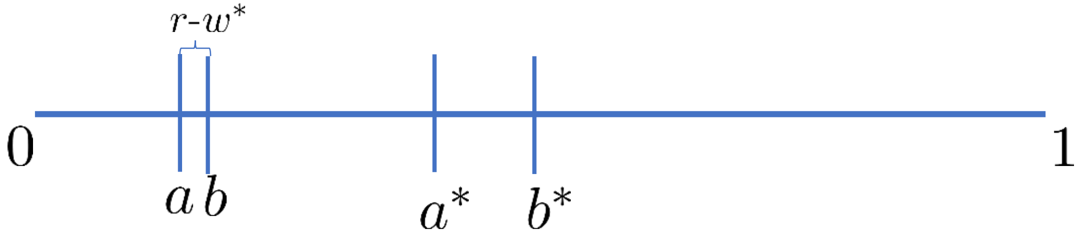
$\text{DIS}(B(f^*, r)) := \{x \in \mathcal{X} : \exists f, f' \in B(f^*, r), f(x) \neq f'(x)\}$

Disagreement coefficient:

$$\theta = \sup_{r > \epsilon} \frac{P_X(\text{DIS}(B(f^*, r)))}{r}$$

Example: **Intervals**, P_X Uniform(0, 1)

$f(x) = \mathbb{I}[a \leq x \leq b]$



$w^* := b^* - a^*$

If $r > w^*$,

$\text{DIS}(B(f^*, r)) = \mathcal{X}$

$P_X(\text{DIS}(B(f^*, r))) = 1$

Sample Complexity Analysis

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

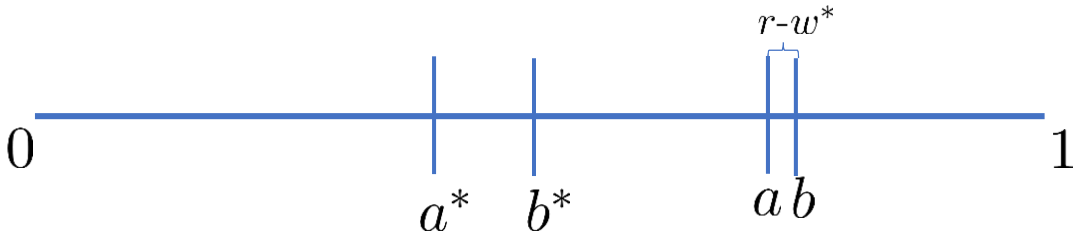
$\mathrm{DIS}(\mathrm{B}(f^*, r)) := \{x \in \mathcal{X} : \exists f, f' \in \mathrm{B}(f^*, r), f(x) \neq f'(x)\}$

Disagreement coefficient:

$$\theta = \sup_{r > \epsilon} \frac{P_X(\mathrm{DIS}(\mathrm{B}(f^*, r)))}{r}$$

Example: **Intervals**, P_X Uniform$(0,1)$

$f(x) = \mathbb{I}[a \leq x \leq b]$



$w^* := b^* - a^*$

If $r > w^*$,

$\mathrm{DIS}(\mathrm{B}(f^*, r)) = \mathcal{X}$

$P_X(\mathrm{DIS}(\mathrm{B}(f^*, r))) = 1$

Sample Complexity Analysis

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

$\mathrm{DIS}(\mathrm{B}(f^*, r)) := \{x \in \mathcal{X} : \exists f, f' \in \mathrm{B}(f^*, r), f(x) \neq f'(x)\}$

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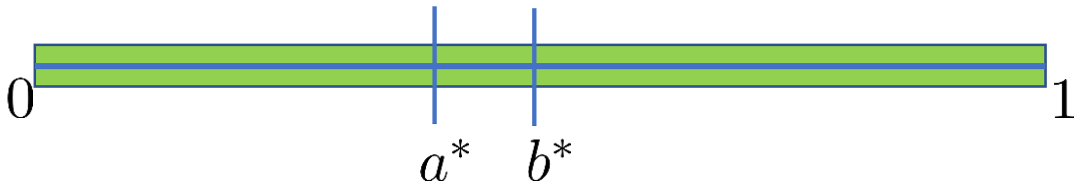
Example: **Intervals**, P_X Uniform$(0, 1)$

$f(x) = \mathbb{I}[a \leq x \leq b]$



$w^* := b^* - a^*$

If $\boldsymbol{r > w^*}$,

$\mathrm{DIS}(\mathrm{B}(f^*, r)) = \mathcal{X}$

$P_X(\mathrm{DIS}(\mathrm{B}(f^*, r))) = 1$

Sample Complexity Analysis

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

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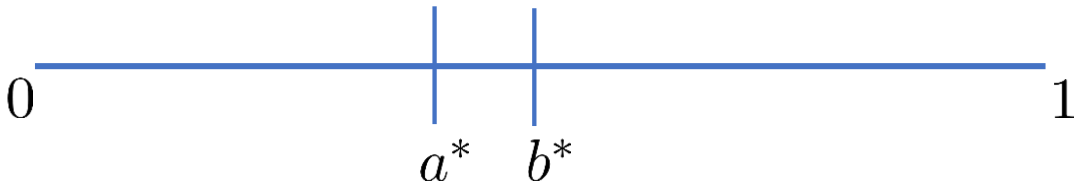
Example: **Intervals**, P_X Uniform$(0, 1)$

$f(x) = \mathbb{I}[a \leq x \leq b]$



$w^* := b^* - a^*$

If $r < w^*$, $P_X(\mathrm{DIS}(\mathrm{B}(f^*, r))) = 4r$

If $r > w^*$, $P_X(\mathrm{DIS}(\mathrm{B}(f^*, r))) = 1$

$\Rightarrow \theta \leq \max\{4, \frac{1}{w^*}\}$

Sample Complexity Analysis

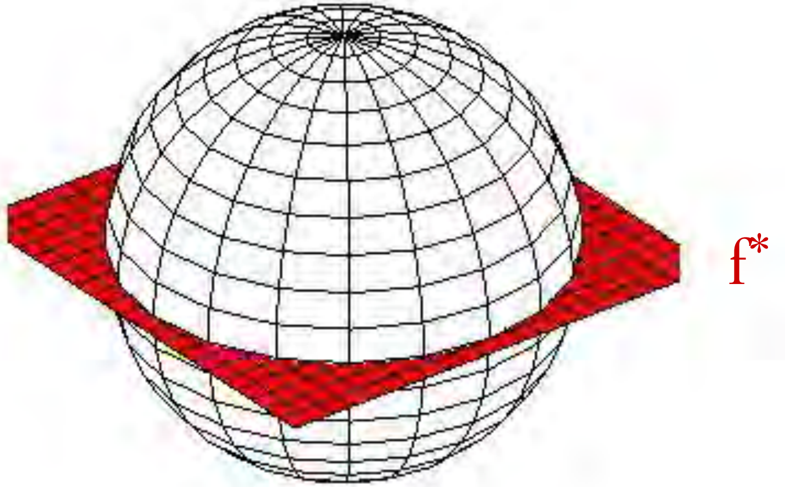
Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

$\mathrm{DIS}(\mathrm{B}(f^*, r)) := \{x \in \mathcal{X} : \exists f, f' \in \mathrm{B}(f^*, r), f(x) \neq f'(x)\}$

Disagreement coefficient:

$$\theta = \sup_{r > \epsilon} \frac{P_X(\mathrm{DIS}(\mathrm{B}(f^*, r)))}{r}$$

Example: homog. linear separators (bias 0), n dimensions, uniform P_X on sphere.



Sample Complexity Analysis

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

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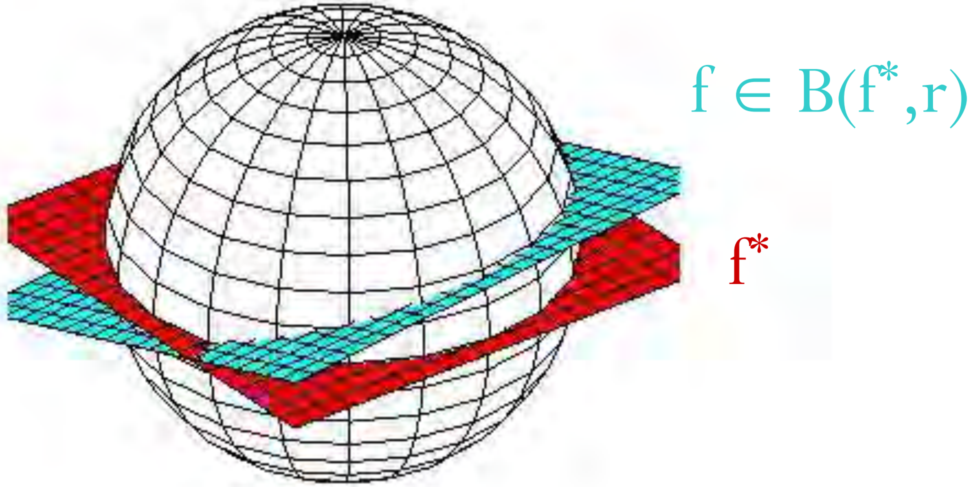

Sample Complexity Analysis

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

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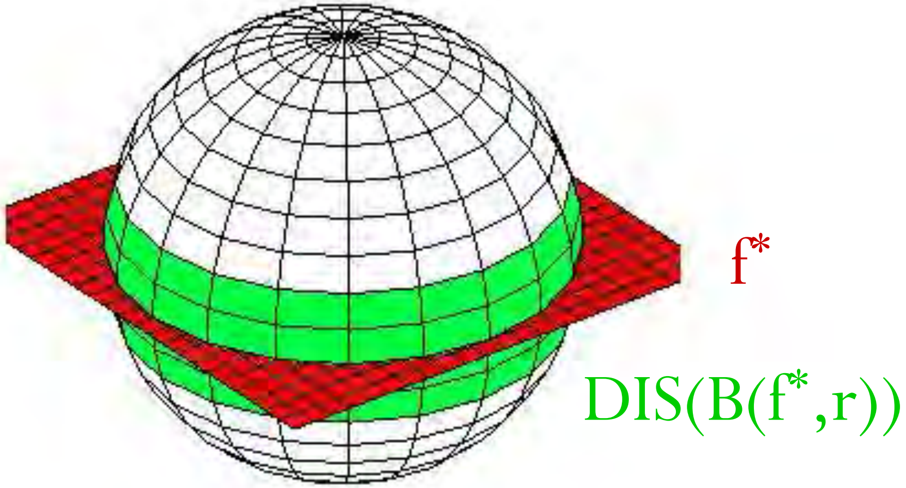

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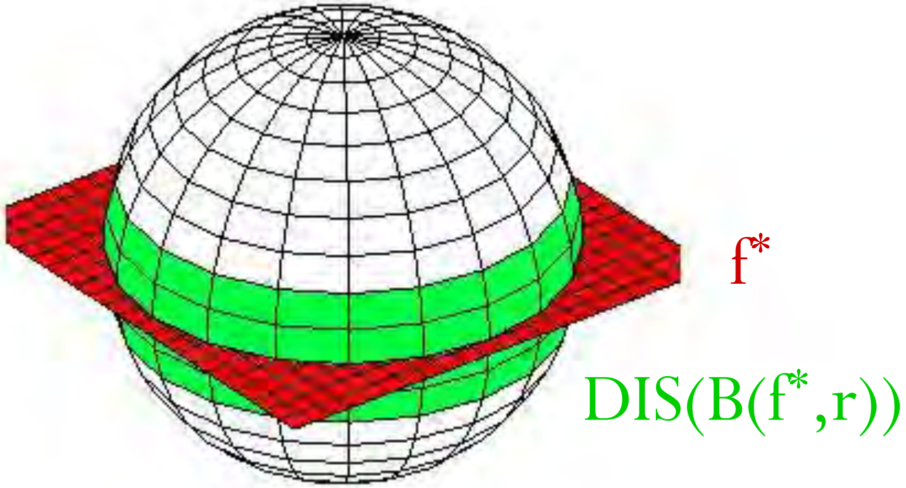
Example: homog. linear separators (bias 0), n dimensions, uniform P_X on sphere.



Some geometry $\Rightarrow$ for small r,

$P_X(DIS(B(f^*, r))) \propto \sqrt{n} r$.

$\Rightarrow \qquad \boldsymbol{\theta \propto \sqrt{n}}$.

Sample Complexity Analysis

Bounded Noise assumption: (aka Massart noise)

$$\exists \beta < 1/2 \text{ s.t. } P(Y \neq f^*(X)|X) \leq \beta \text{ everywhere}$$

	Sample Complexity: $R(\hat{f}) \leq R(f^*) + \epsilon$	Excess Error: n labels
Passive	$\frac{d}{\epsilon}$	$\frac{d}{n}$
Active	$d\theta \log(\frac{1}{\epsilon})$	$e^{-n/d\theta}$

Sample Complexity Analysis

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
2. **label** points in $Q = \mathrm{DIS}(\mathcal{H}) \cap S$
3. **optimize** $\hat{f} \leftarrow \operatorname*{argmin}_{f \in \mathcal{H}} \hat{R}_Q(f)$
4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$.

output final $\hat{f}$

Theorem: $P(Y \neq f^*(X)|X) \leq \beta$. $R(\hat{f}) \leq R(f^*) + \epsilon$ with

$$\#\ \text{labels} \approx d\theta \log(\tfrac{1}{\epsilon}).$$

Proof Sketch:

Round t, all $f \in \mathcal{H}$ **agree** on pts in $S \setminus Q$

Roughly, that means Step 4 only keeps f with
$R(f) - R(f^*) \lesssim \sqrt{P_X(f \neq f^*) \frac{d}{2^t}}$

$\Rightarrow$ surviving f after round t have $R(f) - R(f^*) \lesssim \frac{d}{2^t}$
$\Rightarrow t \gtrsim \log(\frac{d}{\epsilon})$ suffices

Also $\Rightarrow$ after round $t-1$, $\mathcal{H} \subseteq \mathrm{B}(f^*, d/2^{t-1})$

$$\Rightarrow |Q| \lesssim P_X(\mathrm{DIS}(\mathrm{B}(f^*, d/2^{t-1})))|S| \leq \theta \frac{d}{2^{t-1}} |S| = \theta d2$$

$$\sum_{t=1}^{\log(d/\epsilon)} \theta d = \theta d \log(\tfrac{d}{\epsilon}) \qquad \square$$

Sample Complexity Analysis

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
2. **label** points in $Q = \mathrm{DIS}(\mathcal{H}) \cap S$
3. **optimize** $\hat{f} \leftarrow \operatorname*{argmin}_{f \in \mathcal{H}} \hat{R}_Q(f)$
4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$

output final $\hat{f}$

Bounded noise:

$$R(f) - R(f^*) = \int_{f \neq f^*} (P(Y = f^*(X)|X) - P(Y \neq f^*(X)|X)) \mathrm{d}P_X$$
$$\geq (1 - 2\beta) P_X(f \neq f^*)$$

Theorem: $P(Y \neq f^*(X)|X) \leq \beta$. $R(\hat{f}) \leq R(f^*) + \epsilon$ with

$$\#\text{ labels} \approx d\theta \log(\tfrac{1}{\epsilon}).$$

Proof Sketch:

Round t, all $f \in \mathcal{H}$ **agree** on pts in $S \setminus Q$

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Also $\Rightarrow$ after round $t-1$, $\mathcal{H} \subseteq \mathrm{B}(f^*, d/2^{t-1})$

$\Rightarrow |Q| \lesssim P_X(\mathrm{DIS}(\mathrm{B}(f^*, d/2^{t-1})))|S| \leq \theta \frac{d}{2^{t-1}} |S| = \theta d 2$

$$\sum_{t=1}^{\log(d/\epsilon)} \theta d = \theta d \log(\tfrac{d}{\epsilon})$$

□

Sample Complexity Analysis

Agnostic Learning: (no assumptions)

Denote $\beta = R(f^*)$

	Sample Complexity: $R(\hat{f}) \leq R(f^*) + \epsilon$	Excess Error: n labels
Passive	$d\frac{\beta}{\epsilon^2}$	$\sqrt{\frac{d\beta}{n}}$
Active	$d\theta\frac{\beta^2}{\epsilon^2}$	$\sqrt{\frac{d\beta^2\theta}{n}}$

Sample Complexity Analysis

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
2. **label** points in $Q = \mathrm{DIS}(\mathcal{H}) \cap S$
3. **optimize** $\hat{f} \leftarrow \operatorname*{argmin}_{f \in \mathcal{H}} \hat{R}_Q(f)$
4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f})\frac{d}{|Q|}}$.

output final $\hat{f}$

Theorem: $\beta = R(f^*)$. $R(\hat{f}) \leq R(f^*) + \epsilon$ with

$$\#\text{ labels} \approx d\theta\frac{\beta^2}{\epsilon^2}.$$

Proof Sketch:

Round t, all $f \in \mathcal{H}$ **agree** on pts in $S \setminus Q$

Roughly, that means Step 4 only keeps f with

$R(f) - R(f^*) \lesssim \sqrt{P_X(f \neq f^*)\frac{d}{2^t}}$

$\Rightarrow$ surviving f after round t have $R(f) - R(f^*) \lesssim \sqrt{\beta\frac{d}{2^t}} + \frac{d}{2^t}$

(Roughly) $\sqrt{\beta\frac{d}{2^t}}$

$\Rightarrow t \gtrsim \log(d\frac{\beta}{\epsilon^2})$ suffices

Also $\Rightarrow$ after round $t-1$, $\mathcal{H} \subseteq \mathrm{B}\left(f^*, 2\beta + \sqrt{\beta\frac{d}{2^{t-1}}}\right) \subseteq \mathrm{B}(f^*, 3\beta)$ (for large t)

$\Rightarrow |Q| \lesssim P_X(\mathrm{DIS}(\mathrm{B}(f^*, 3\beta)))|S| \lesssim \theta\beta|S| = \theta\beta 2^t$

$$\sum_{t=1}^{\log(d\beta/\epsilon^2)} \theta\beta 2^t \sim \theta d\frac{\beta^2}{\epsilon^2}$$

□

Sample Complexity Analysis

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

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4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f})\frac{d}{|Q|}}$

output final $\hat{f}$

$$P_X(f \neq f^*) \leq R(f) + R(f^*) = 2\beta + R(f) - R(f^*)$$

Theorem: $\beta = R(f^*)$. $R(\hat{f}) \leq R(f^*) + \epsilon$ with

$$\#\ \text{labels} \approx d\theta\frac{\beta^2}{\epsilon^2}.$$

Proof Sketch:

Round t, all $f \in \mathcal{H}$ **agree** on pts in $S \setminus Q$

Roughly, that means Step 4 only keeps f with
$R(f) - R(f^*) \lesssim \sqrt{P_X(f \neq f^*)\frac{d}{2^t}}$

$\Rightarrow$ surviving f after round t have $R(f) - R(f^*) \lesssim \sqrt{\beta\frac{d}{2^t}} + \frac{d}{2^t}$

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Sample Complexity Analysis

When is θ small?

- Linear separators, P_X has a density,
  f^* boundary intersects interior of support
  $\Rightarrow$ **$\boldsymbol{\theta}$ bounded**

- Linear separators, P_X has a density
  $\Rightarrow \boldsymbol{\theta \ll \frac{1}{\epsilon}}$

- $\mathcal{H}$ smoothly-parametrized model,
  P_X "regular" density w/ compact support,
  other technical conditions on $\mathcal{H}$
  $\Rightarrow$ **$\boldsymbol{\theta \propto \#}$ parameters for $\boldsymbol{\mathcal{H}}$**

- $\cdots$

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- $\ldots$

Lots more →

Stopping Criterion

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
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4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$.

output final $\hat{f}$

Stopping criteria:

- Any-time
- Label budget
- Run out of unlabeled data
- Check $\max_{f \in \mathcal{H}} \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}} < \epsilon$

Simpler Agnostic Active Learning

Hsu (2010,...)

$Q \leftarrow \{\}$

for $m = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** a random point x
2. **optimize** $\forall y,\ \hat{f}_y \leftarrow \underset{f \in \mathcal{H}: f(x)=y}{\operatorname{argmin}} \ \hat{R}_Q(f)$
3. if $|\hat{R}_Q(\hat{f}_+) - \hat{R}_Q(\hat{f}_-)| \leq \sqrt{\hat{P}_Q(\hat{f}_- \neq \hat{f}_+)\frac{d}{|Q|}}$

   then **label** x, add it to Q

output $\hat{f} = \underset{f \in \mathcal{H}}{\operatorname{argmin}}\, \hat{R}_Q(f)$

- Roughly same sample complexity as A^2.
- Can implement as a **reduction** to ERM.
- In practice, replace ERM with any passive learner.

Surrogate Loss

Hanneke & Yang (2012)

$Q \leftarrow \{\}$

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1. **sample** a random point x
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3. if $|\hat{R}_Q(\hat{f}_+) - \hat{R}_Q(\hat{f}_-)| \le \sqrt{\hat{P}_Q(\hat{f}_- \neq \hat{f}_+)\frac{d}{|Q|}}$

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Consider learner that minimizes a **surrogate loss**
$\ell : \mathbb{R} \times \{-1, +1\} \to \mathbb{R}_+$
(e.g., hinge loss, squared loss, exponential loss, ...)

Now $\mathcal{H}$ is **real-valued** functions
$\hat{R}^{\ell}_Q(f) = \frac{1}{|Q|} \sum_{(x,y) \in Q} \ell(f(x), y)$

Theorem: Bounded noise, plus strong assumptions on $\mathcal{H}$,ℓ,P
still get $R(\hat{f}) \le R(f^*) + \epsilon$ with # labels

$$\approx \theta d \log(\tfrac{1}{\epsilon})$$

Importance-Weighted Active Learning

Beygelzimer, Dasgupta, Langford (2009)

$Q \leftarrow \{\}$

for $m = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** a random point x
2. **set** sampling probability p_x
3. **flip** coin with prob p_x of heads
4. if heads, **label** x, add to Q with weight $1/p_x$

output $\hat{f} = \operatorname{argmin}_{f \in \mathcal{H}} \hat{R}_Q(f)$ (weighted loss)

Use importance weights to stay **unbiased**:
$\mathbb{E}[\hat{R}_Q(f)] = R(f)$

Now Q set of triples (x, y, w)

$$\hat{R}_Q(f) = \frac{1}{|Q|} \sum_{(x,y,w) \in Q} w \mathbb{I}[f(x) \neq y]$$

- **Any** choice of Step 2 (setting p_x) is fine (just p_x not too small, else high variance)

- Can set p_x in a way to recover A^2 sample complexity

$$p_x = \mathbb{I}\left[\, |\hat{R}_Q(\hat{f}_+) - \hat{R}_Q(\hat{f}_-)| \leq \sqrt{\hat{P}_Q(\hat{f}_+ \neq \hat{f}_-)\frac{d}{|Q|}}\,\right]$$

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- In practice, replace ERM with any passive learner (e.g., ERM with a surrogate loss)
- (approx) implementation in **Vowpal Wabbit** library

Questions?

Further reading:

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Part 3: Beyond Disagreement-Based Active Learning – Current Directions

- Subregion-Based Active Learning
- Margin-Based Active Learning: Linear Separators
- Shattering-Based Active Learning
- Distribution-Free Analysis, Optimality
- TicToc: Adapting to Heterogeneous Noise
- Tsybakov Noise

Tutorial on Active Learning: Theory to Practice

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Thirty-sixth International Conference on Machine Learning

Subregion-Based Active Learning

Zhang & Chaudhuri, 2014

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Subregion-based Active Learning

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
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Instead, pick **region $\mathcal{R}_{\epsilon'}(\mathcal{H})$** s.t.
$\forall f, f' \in \mathcal{H},\ P_X(x \notin \mathcal{R}_{\epsilon'}(\mathcal{H}) : f(x) \neq f'(x)) \leq \epsilon'$.

Pick ϵ' carefully each round,
$R(\hat{f}) - R(f^*) \leq \epsilon$ at end

e.g., Bounded noise: $\epsilon' \propto d2^{-t}$

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$\forall f, f' \in \mathcal{H},\ P_X(x \notin \mathcal{R}_{\epsilon'}(\mathcal{H}) : f(x) \neq f'(x)) \leq \epsilon'$.

$$\varphi_c := \sup_{r > \epsilon} \frac{P_X(\mathcal{R}_{r/c}(\mathrm{B}(f^*, r)))}{r}$$

Theorem: with **Bounded noise**,
$R(\hat{f}) \leq R(f^*) + \epsilon$ using # labels

$$\approx \varphi_c d \log\left(\frac{1}{\epsilon}\right)$$

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Agnostic case: $\varphi'_c := \sup_{r > \epsilon} \frac{P_X(\mathcal{R}_{r/c}(\mathrm{B}(f^*, 2\beta + r)))}{2\beta + r}$

Theorem:
$R(\hat{f}) \leq R(f^*) + \epsilon$ using # labels

$$\approx \varphi'_c d \frac{\beta^2}{\epsilon^2}$$

Subregion-Based Active Learning

Zhang & Chaudhuri, 2014

How to find such an $\mathcal{R}_{\epsilon'}(\mathcal{H})$?

- $\mathcal{R}_{\epsilon'}(\mathcal{H}) = \mathrm{DIS}(\mathcal{H})$ works
- Empirically (Zhang & Chaudhuri, 2014)
- Nice structure: e.g., **Linear separators**

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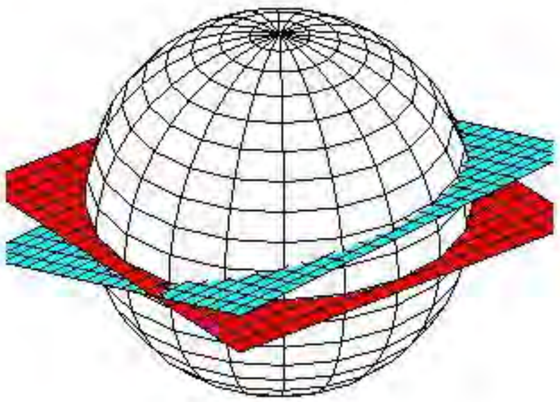
Uniform P_X on d-dim sphere

For $w \in \mathrm{B}(w^*, r)$, **project** to Span(w,w^*)



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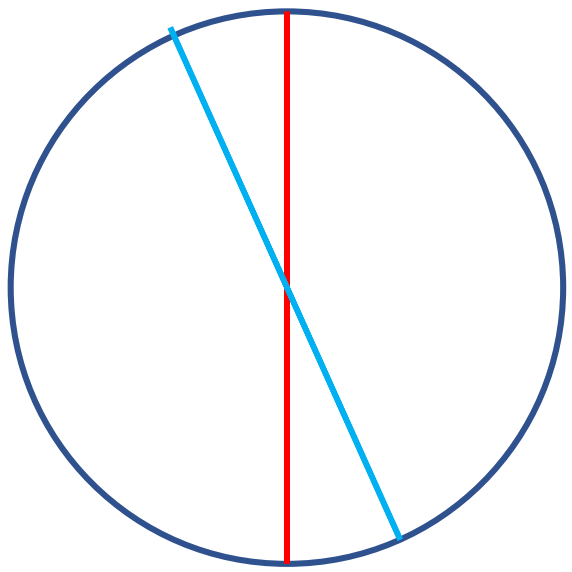
For $w \in \mathrm{B}(w^*, r)$, **project** to Span(w,w^*)

Most projected prob mass toward middle



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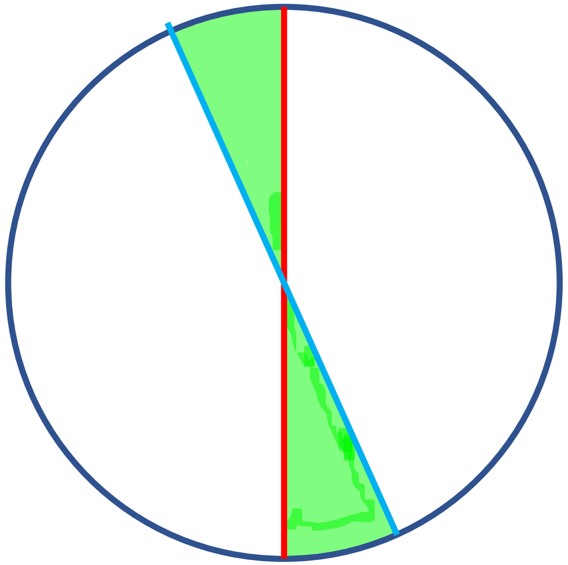

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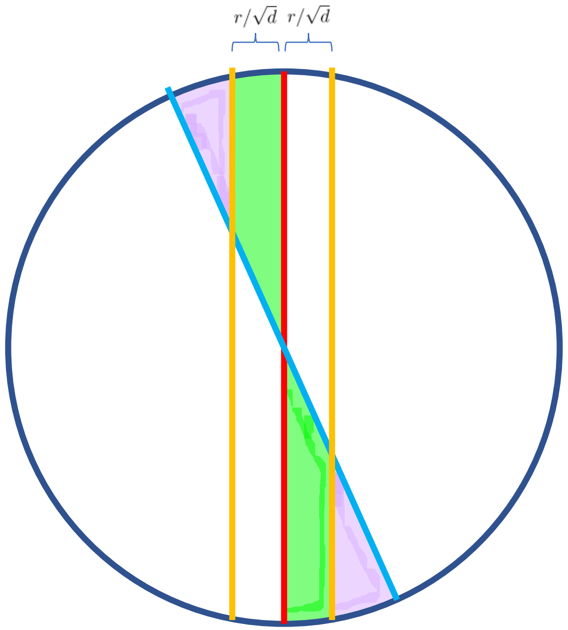

DIS($\{w, w^*\}$) in slab of width $\approx r$

Most of its prob in slab of width $\approx r/\sqrt{d}$

Subregion-Based Active Learning

How to find such an $\mathcal{R}_{\epsilon'}(\mathcal{H})$?

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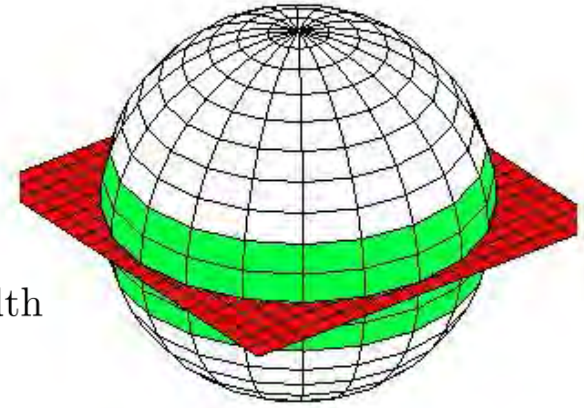
$\mathrm{DIS}(\mathrm{B}(f^*, r)) =$
slab of width $\approx r$

Take $\mathcal{R}_{r/c}(\mathrm{B}(f^*, r)) =$
slab of width $\approx r/\sqrt{d}$

Prob in slab $\approx \sqrt{d} \times$ width

$\Rightarrow \varphi_c \leq$ constant



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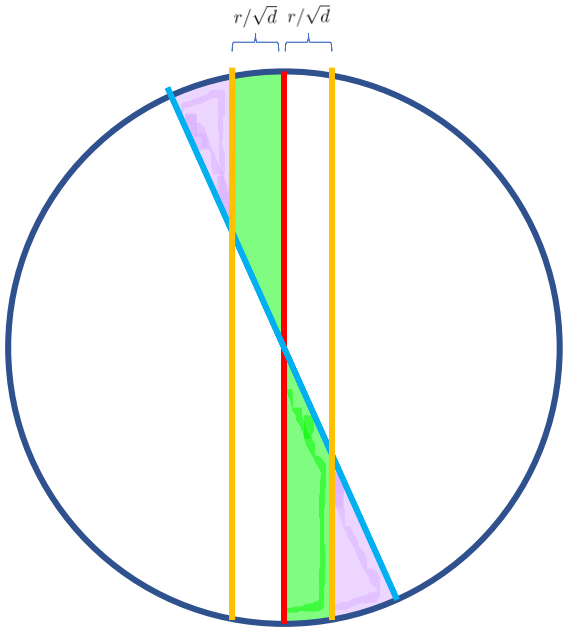

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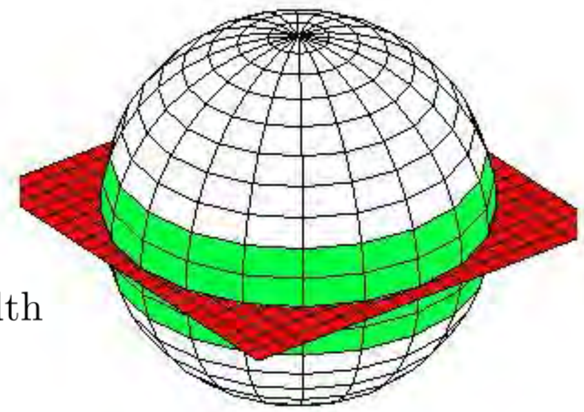

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$$\varphi_c := \sup_{r > \epsilon} \frac{P_X(\mathcal{R}_{r/c}(\mathrm{B}(f^*, r)))}{r}$$

Theorem: with **Bounded noise**,
$R(\hat{f}) \leq R(f^*) + \epsilon$ using # labels
$\approx \varphi_c d \log\left(\frac{1}{\epsilon}\right)$

$\Rightarrow$ # labels $\approx d \log(\frac{1}{\epsilon})$ suffice

Recall:
Passive $\approx \frac{d}{\epsilon}$

Comparison:
Recall $\theta \approx \sqrt{d}$
$\Rightarrow A^2$ # labels $\approx d^{3/2} \log(\frac{1}{\epsilon})$

Margin-Based Active Learning

(Balcan, Broder, Zhang, 2007; ...)

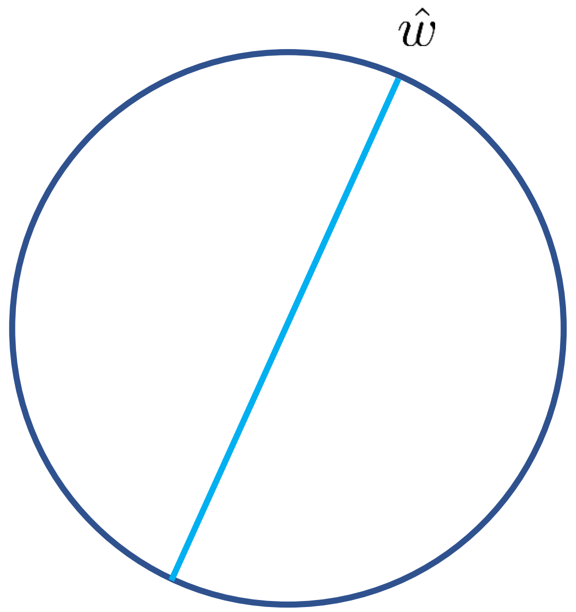
Margin-based Active Learning

Initialize $\hat{w}$

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** $d2^t$ unlabeled points S
2. **label** points in $Q =$ all $x \in S$ s.t. $<\hat{w}, x> \;\; \leq \;\; c2^{-t}/\sqrt{d}$
3. **optimize** $\hat{w} \leftarrow \underset{w:\|w-\hat{w}\| \leq c' 2^{-t}}{\text{argmin}} \hat{R}_Q(w)$

output final $\hat{w}$



Uniform P_X on d-dim sphere

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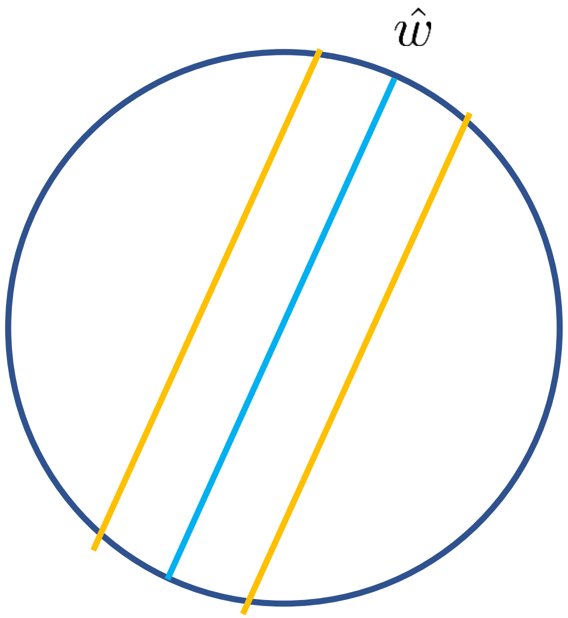

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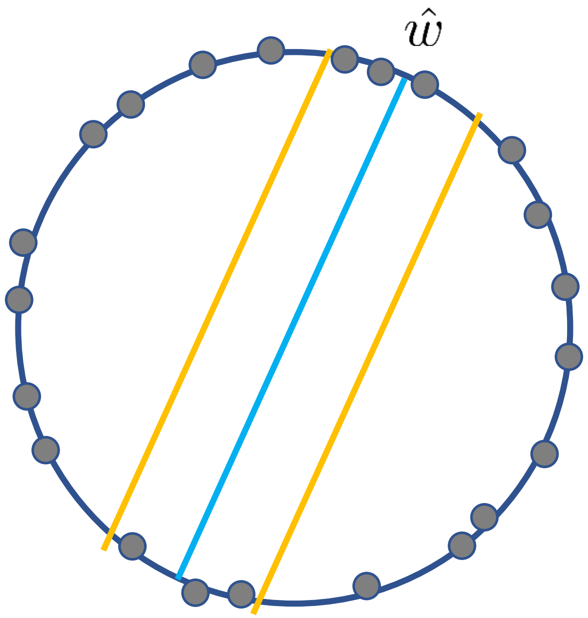

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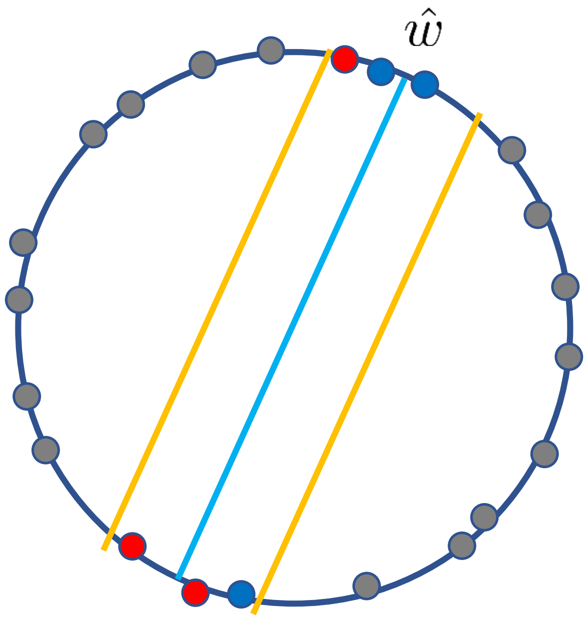

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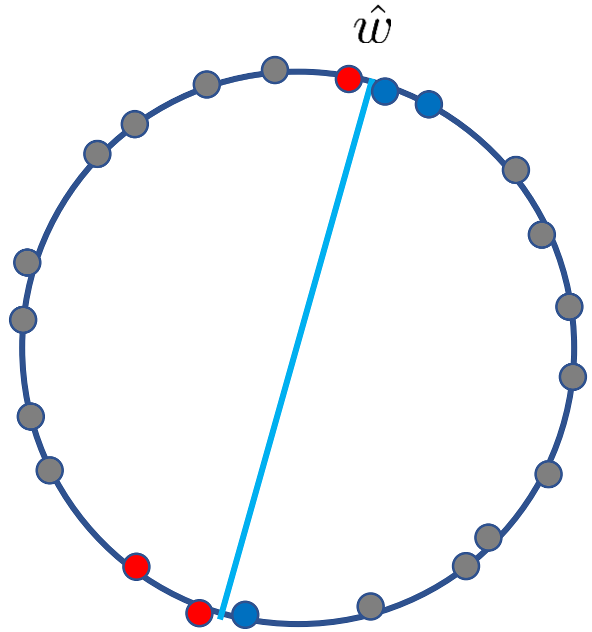

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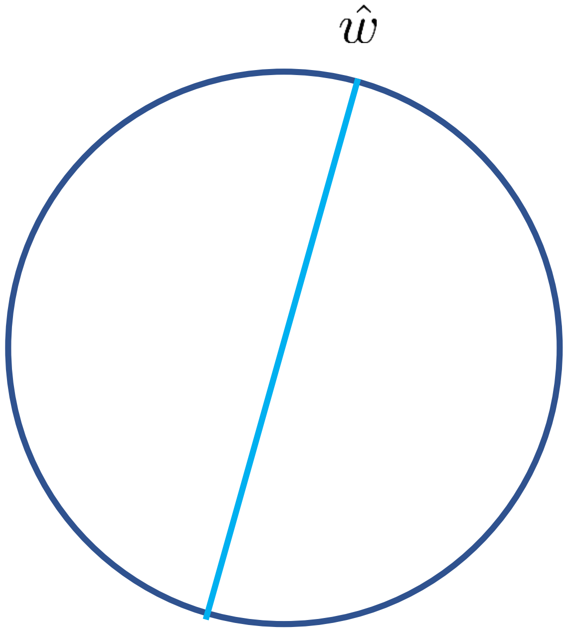


Uniform P_X on d-dim sphere

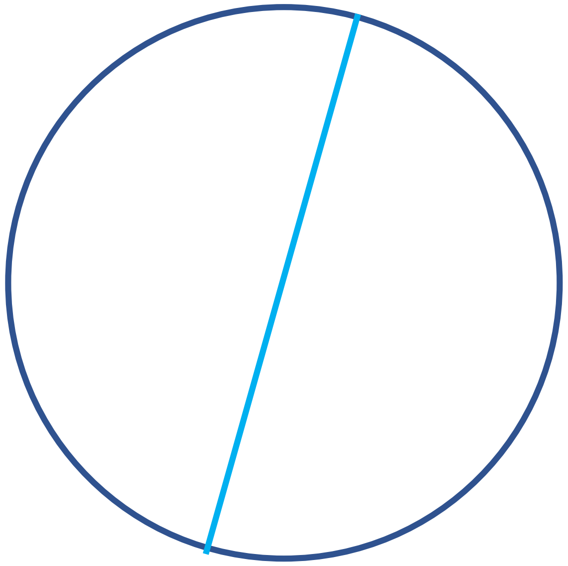
Theorem: with **Bounded noise**,
$R(\hat{f}) \leq R(f^*) + \epsilon$ using # labels
$\approx d \log\left(\frac{1}{\epsilon}\right)$

(also works for isotropic log-concave distributions)

Computational Efficiency

(Awasthi, Balcan, Long, 2014,...)

Uniform P_X on d-dim sphere



Efficient Alg

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output final $\hat{w}$

Surrogate loss

$$\ell_t(<w, x>, y) \approx \max\{1 - 2^t\sqrt{d}(y<w, x>), 0\}$$

Hinge loss slope **changes** each round

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and running in polynomial time

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and running in polynomial time

Theorem: with **Agnostic** case,
$R(\hat{f}) \leq CR(f^*)$ **in polynomial time**

(was first alg. known to achieve these; even passively)

(also works for isotropic log-concave distributions)

Up Next:
Shattering-Based Active Learning

Shattering-Based Active Learning

(Hanneke, 2009, 2012)

Recall: $\mathcal{H}$ **shatters** $x_1, \ldots, x_k$ if all 2^k classifications realized by $\mathcal{H}$

DIS($\mathcal{H}$) checks for shattering 1 point.

Idea: Generalize to shattering ≥ 1 points.

Shattering-Based Active Learning

(Hanneke, 2009, 2012)

Recall: $\mathcal{H}$ **shatters** $x_1, \ldots, x_k$ if all 2^k classifications realized by $\mathcal{H}$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
2. **label** points in $Q = \mathrm{DIS}(\mathcal{H}) \cap S$
3. **optimize** $\hat{f} \leftarrow \operatorname*{argmin}_{f \in \mathcal{H}} \hat{R}_Q(f)$
4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$.

output final $\hat{f}$

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Shattering-based Active Learning

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Idea: Generalize to shattering ≥ 1 points.

Denote $\mathcal{H}_{x,y} := \{h \in \mathcal{H} : h(x) = y\}$

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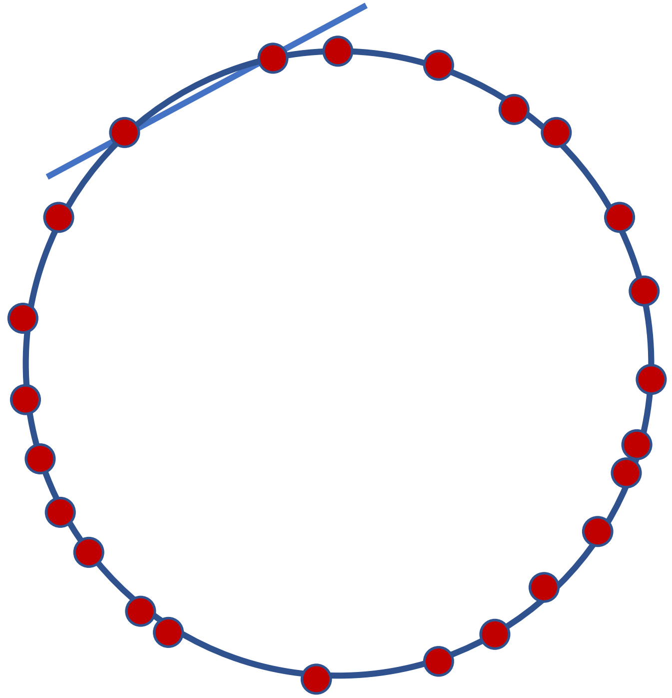
Denoting $\mathcal{H}_{x,y} := \{h \in \mathcal{H} : h(x) = y\}$

Example: Linear separators, Uniform P_X on circle

Suppose true labels are **all** -1

$\mathrm{DIS}(\mathcal{H}) =$ **entire circle**



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Suppose true labels are **all** $\mathbf{-1}$

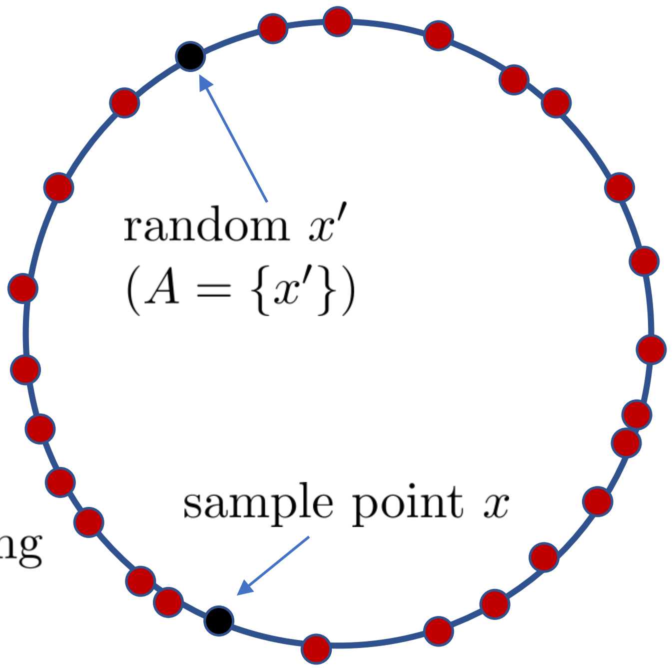
$\text{DIS}(\mathcal{H})$ = **entire circle**

Try $k = 1$

Given sample x
Rand x' probably not close

Can't shatter $\{x, x'\}$ without a lot of points wrong

So won't query x



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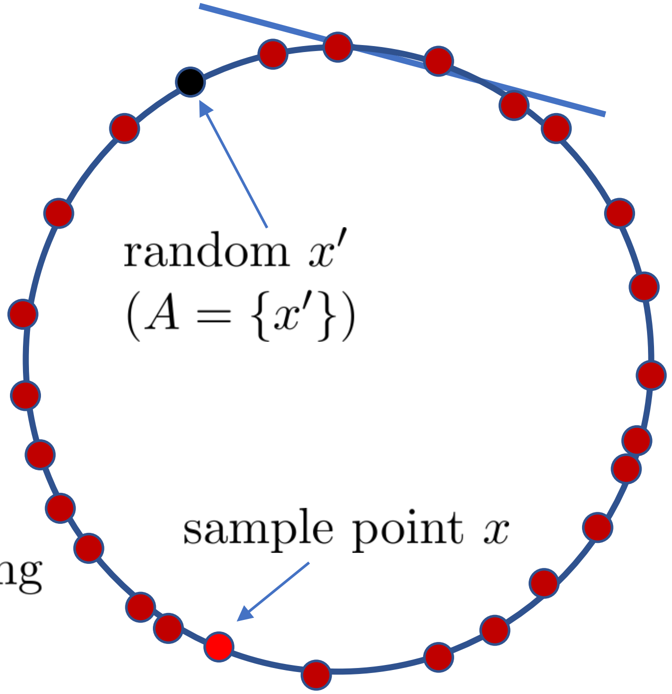

$\mathrm{DIS}(\mathcal{H}_{x,-1})$ still entire circle (minus x)

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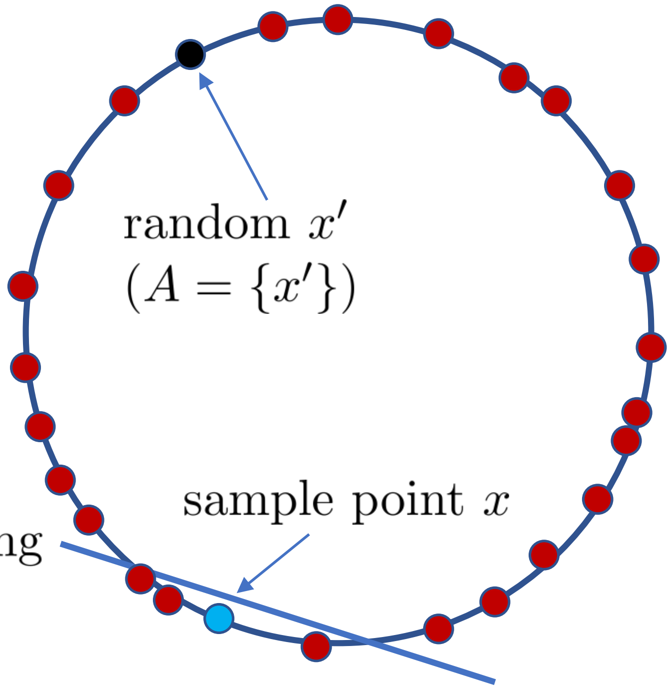

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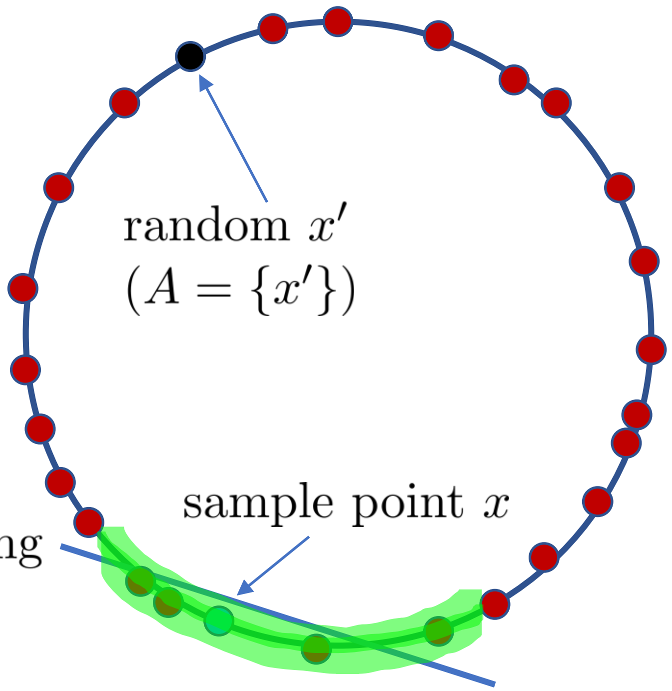

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Generally, need to try various k and pick one (See the papers)

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(See the papers)

$$\theta^{(k)} := \sup_{r > \epsilon} \frac{P_X^k(A \in \mathcal{X}^k : \mathrm{B}(f^*, r) \text{ shatters } A)}{r}$$

$$\tilde{d} := \min\left\{k : P_X^k(A \in \mathcal{X}^k : \mathrm{B}(f^*, r) \text{ shatters } A) \xrightarrow[r \to 0]{} 0\right\}$$

$$\tilde{\theta} := \theta^{(\tilde{d})}$$

Theorem: For Bounded noise, $R(\hat{f}) \leq R(f^*) + \epsilon$ with # labels

$$\approx C\tilde{\theta} d \log\left(\tfrac{1}{\epsilon}\right)$$

Note: $\tilde{\theta} \ll \frac{1}{\epsilon}$

Shattering-Based Active Learning

Recall: $\mathcal{H}$ **shatters** $x_1, \ldots, x_k$ if all 2^k classifications realized by $\mathcal{H}$

Shattering-based Active Learning

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
2. **label** points in $Q =$ all $x \in S$ s.t. $P_X^k(A \in \mathcal{X}^k : \mathcal{H} \text{ shatters } A \cup \{x\} | \mathcal{H} \text{ shatters } A) \geq \frac{1}{2}$
3. **add** the remaining points $x \in S$ to Q with label $\hat{y}_x := \operatorname{argmax}_y P_X^k(A \in \mathcal{X}^k : \mathcal{H}_{x,y} \text{ shatters } A | \mathcal{H} \text{ shatters } A)$
4. **optimize** $\hat{f} \leftarrow \operatorname{argmin}_{f \in \mathcal{H}} \hat{R}_Q(f)$
5. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$

output final $\hat{f}$

Denoting $\mathcal{H}_{x,y} := \{h \in \mathcal{H} : h(x) = y\}$

Generally, need to try various k and pick one (See the papers)

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In the example: $\tilde{\theta} = 2$, $\theta = \frac{1}{\epsilon}$

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(may depend on f^*, P_X)

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In the example: $\tilde{\theta} = 2$, $\theta = \frac{1}{\epsilon}$

Up Next:
Distribution-free Analysis

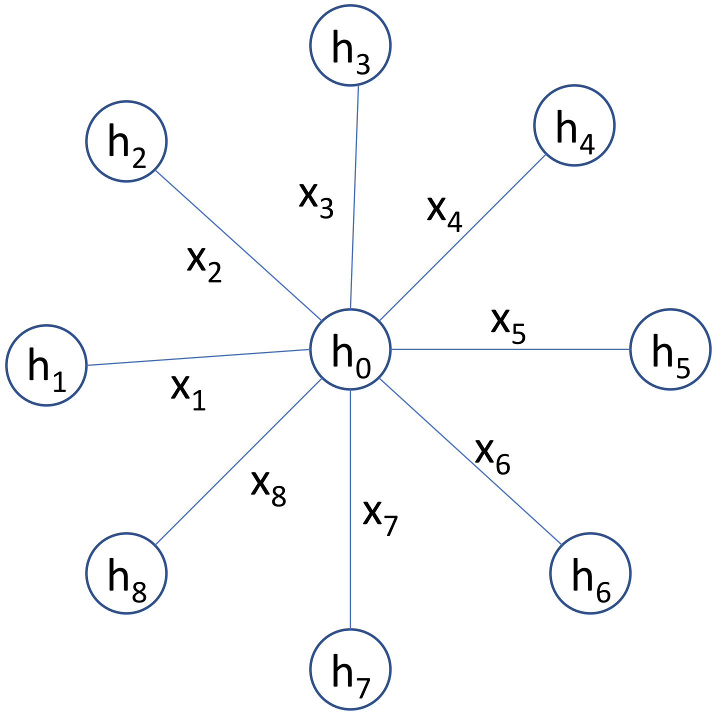
Distribution-Free Analysis

(Hanneke & Yang, 2015)

θ, φ, $\tilde{\theta}$ depend on f^*, P_X.

Can we do sample complexity analysis **without** distribution-dependence?

Definition: The **star number** $\mathfrak{s}$ is the largest k s.t. $\exists h_0, h_1, \ldots, h_k \in \mathcal{H}$, $\exists x_1, \ldots, x_k \in \mathcal{X}$ s.t. $\forall i \in \{1, \ldots, k\}$, $\{x_j : h_i(x_j) \neq h_0(x_j)\} = \{x_i\}$.



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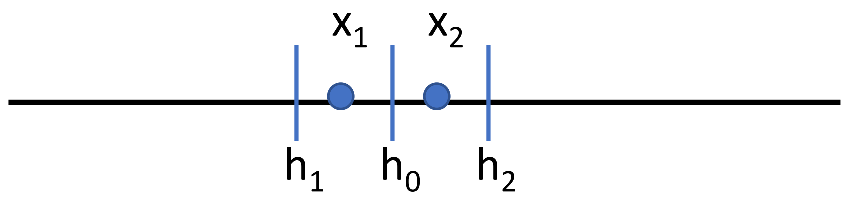
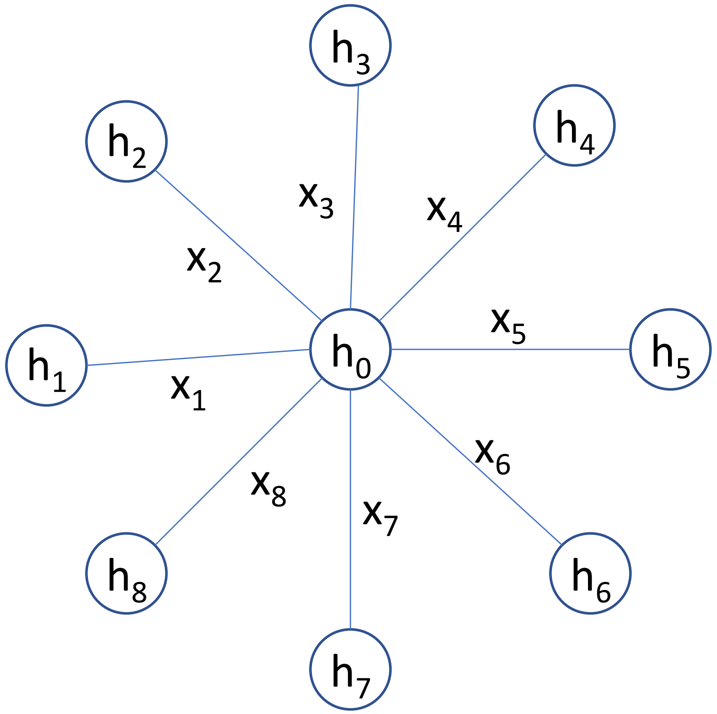
Example: Thresholds: $f(x) = \mathbb{I}[x \geq t]$.

$\mathfrak{s} = 2$.





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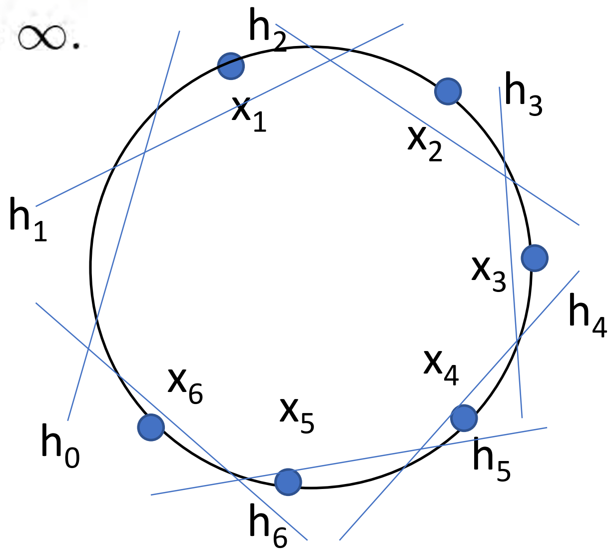
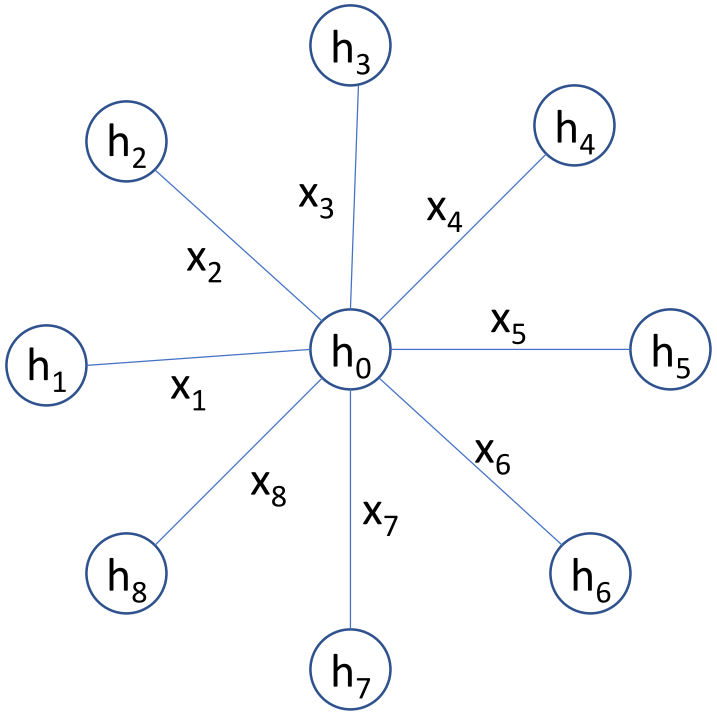
Example: Linear Separators in $\mathbb{R}^n$, $n \geq 2$:

$\mathfrak{s} = \infty$.





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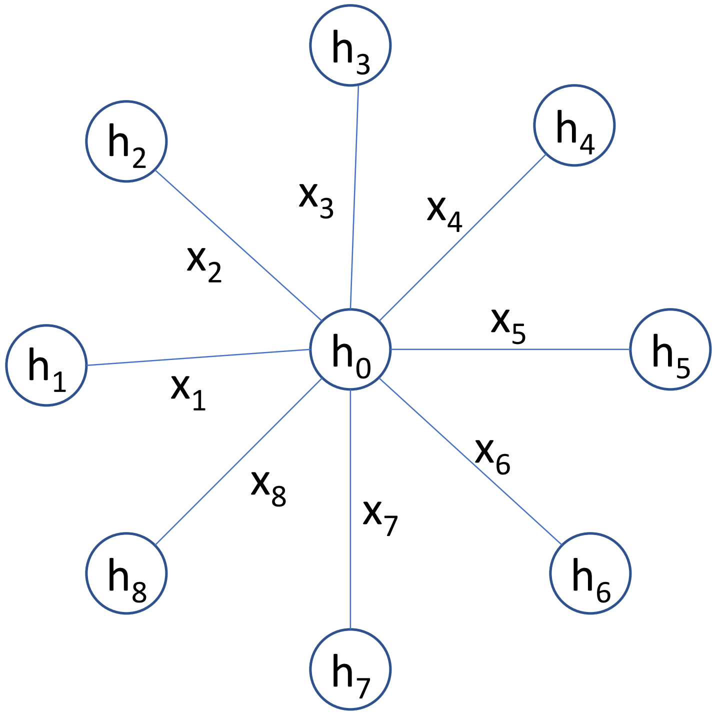
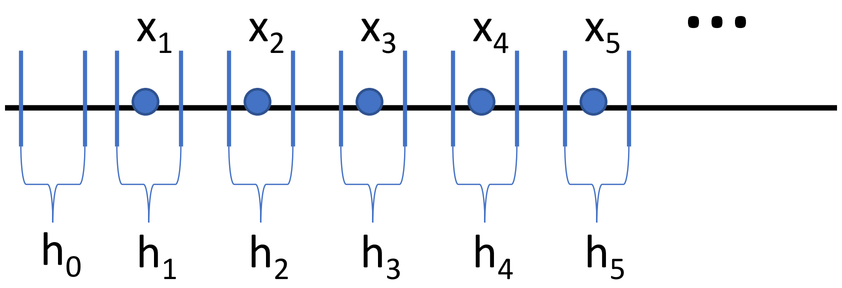


Example: Intervals: $x \mapsto \mathbb{I}[a \leq x \leq b]$

$\mathfrak{s} = \infty$.



Intervals of width w ($b - a = w > 0$) on $\mathcal{X} = [0, 1]$: $\mathfrak{s} \approx \left\lfloor \frac{1}{w} \right\rfloor$.

Distribution-Free Analysis

(Hanneke & Yang, 2015; Hanneke, 2016)

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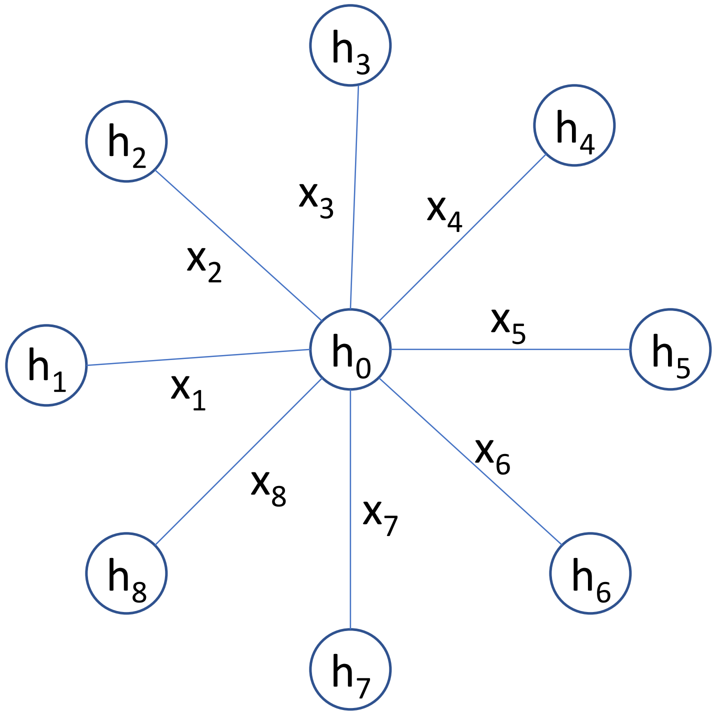

Theorem: $\sup_{P_X} \sup_{f^* \in \mathcal{H}} \theta = \sup_{P_X} \sup_{f^* \in \mathcal{H}} \varphi_c = \sup_{P_X} \sup_{f^* \in \mathcal{H}} \tilde{\theta} = \min\{\mathfrak{s}, \frac{1}{\epsilon}\} =: \mathfrak{s}_\epsilon$

Corollary:

Bounded noise # labels $\approx \mathfrak{s}_\epsilon d \log(\frac{1}{\epsilon})$

Agnostic ($\beta = R(f^*)$) # labels $\approx \mathfrak{s}_\beta d \frac{\beta^2}{\epsilon^2}$

Achieved by A^2

Distribution-Free Analysis

(Hanneke & Yang, 2015; Hanneke, 2016)

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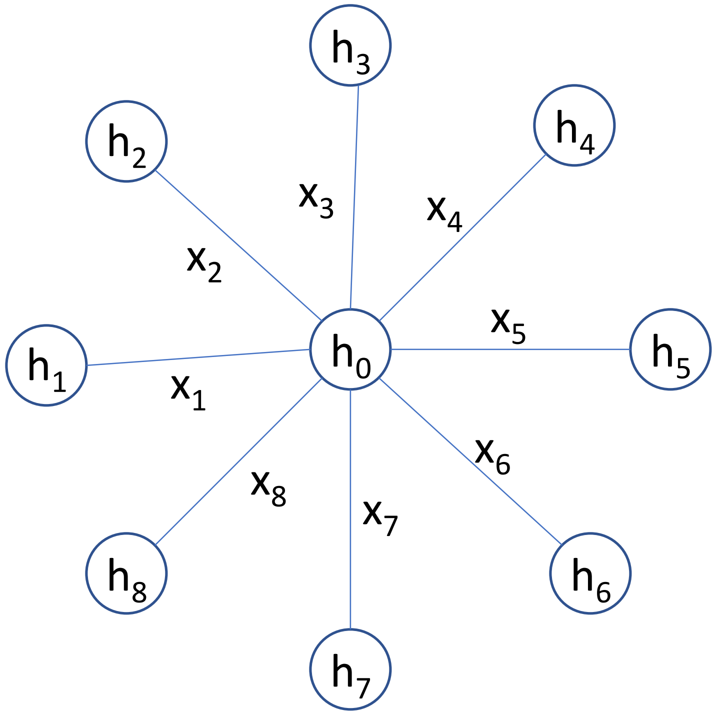
Achieved by A^2

Different alg., Bounded noise
labels $\approx \mathfrak{s}_{\epsilon/d} \log(\frac{1}{\epsilon})$

Near-matching **lower bound**:
$\mathfrak{s}_\epsilon + d \log(\frac{1}{\epsilon})$



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$\mathfrak{s}_\epsilon + d \log(\frac{1}{\epsilon})$

Open Question:

Agnostic ($\beta = R(f^*)$)

labels

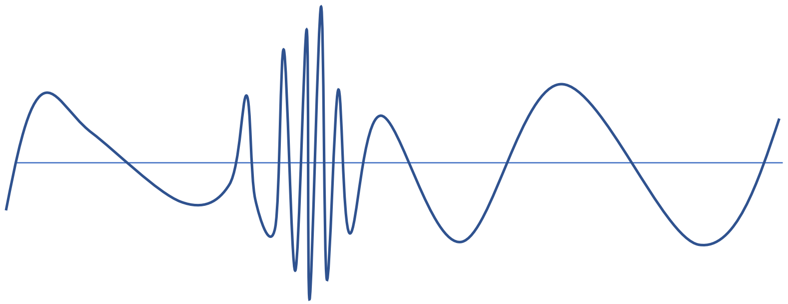
$\approx d\frac{\beta^2}{\epsilon^2} + \mathfrak{s}_{\epsilon/d} \log(\frac{1}{\epsilon})$ **?**

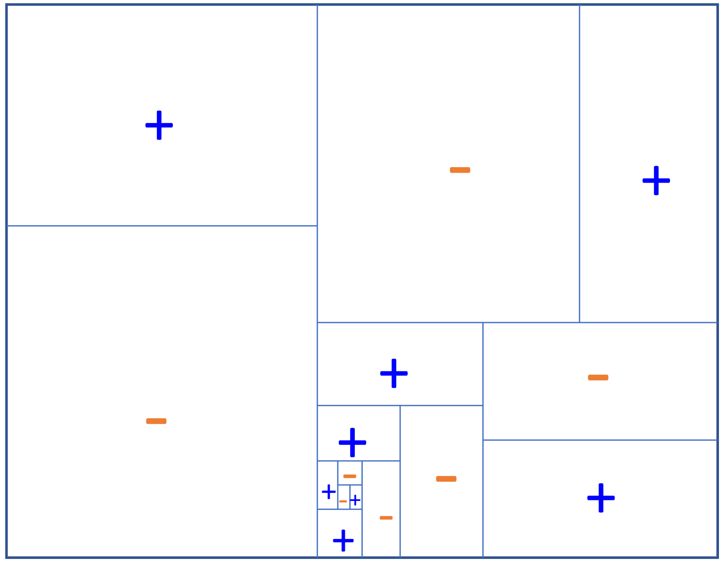
lower bound:

$d\frac{\beta^2}{\epsilon^2} + \mathfrak{s}_\epsilon + d \log(\frac{1}{\epsilon})$

Adapting to Heterogeneous Noise

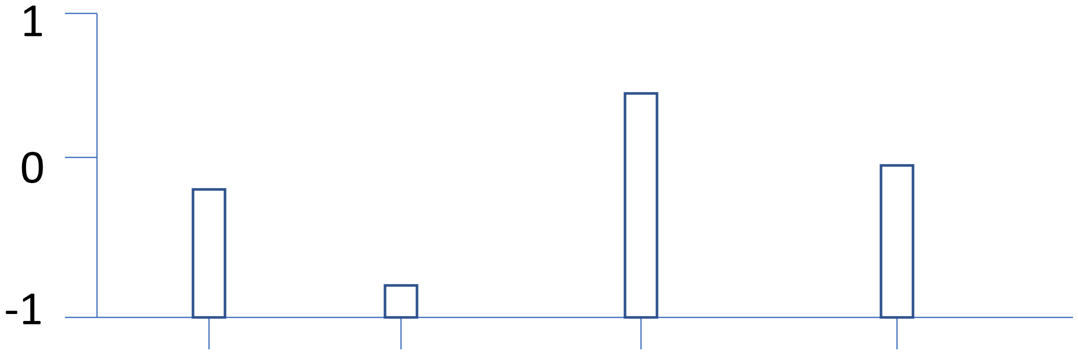
So far: Active learning for spatial heterogeneity of **opt function**:





Also consider: Spatial heterogeneity of **noise**:

$\eta(x) := \mathbb{E}[Y|X = x]$



Active Learning with TicToc

(Hanneke & Yang, 2015)

Algorithm: $\mathbb{A}(n)$
Input: Label budget n
Output: Classifier $\hat{f}_n$.

$\mathbb{L} \leftarrow \{\}$
For $m = 1, 2, \ldots$
  $X_{s_m} \leftarrow \textsc{GetSeed}(\mathbb{L}, m)$
  $\mathcal{L}_m \leftarrow \textsc{TicToc}(X_{s_m}, m)$
  if $\mathcal{L}_m$ exists, $\mathbb{L} \leftarrow \mathbb{L} \cup \{(s_m, \mathcal{L}_m)\}$
  If we've made n queries
    Return $\hat{f}_n \leftarrow \textsc{Learn}(\mathbb{L})$

An active learning alg. (e.g. A^2)

Main new part

A passive learning alg.

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Denote $\eta(x) = \mathbb{E}[Y|X = x]$
Suppose f^* is the **global** optimal function: $f^*(x) = \text{sign}(\eta(x))$

$\textsc{TicToc}(X, m)$:
Query X (or nearby) to try to guess $f^*(X)$
If can figure it out, return that label
If can't figure it out by τ_m queries give up (don't return a label)

Focus queries on less-noisy points.

Double advantage:

- Focusing on the points we actually care about:

$$R(f|x) - R(f^\star|x) = |\eta(x)|\mathbb{I}[f(x) \neq f^\star(x)]$$

(small $|\eta(x)|$ $\Rightarrow$ not much effect on $R(f|x)$ if $f(x) = f^*(x)$ or not).

- And those points require fewer queries to determine $f^\star(X_i)$!

$\sim \frac{1}{\eta(X_i)^2}$ queries
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Theorem: Agnostic ($\beta = R(f^*)$)
and suppose $f^* =$ global best:
\# labels

$$\approx d\frac{\beta^2}{\epsilon^2} + \mathfrak{s}_{\epsilon/d} \log(\tfrac{1}{\epsilon})$$

Confirms agnostic sample complexity conjecture but with extra assumption $f^* =$ global opt.

Near-match lower bound: $d\frac{\beta^2}{\epsilon^2} + \mathfrak{s}_\epsilon + d\log(\frac{1}{\epsilon})$

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Principles of Active Learning

1. Query in dense regions where $\hat{f}$ could disagree a lot with f^*
2. Query in regions with low noise

Tsybakov Noise

The alg. adapts to heterogeneity in the noise.

Let’s try it with a model that explicitly describes heterogeneous noise:

Tsybakov Noise

Tsybakov Noise

(Tsybakov, 2004;
Mammen & Tsybakov 1999)

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Definition: (Tsybakov noise)
$f^\star(x) = \text{sign}(\eta(x))$ and $\exists \alpha \in (0,1)$ s.t. $\forall \tau > 0$,

$$P_X(x : |\eta(x)| \leq \tau) \lesssim \tau^{\frac{\alpha}{1-\alpha}}.$$

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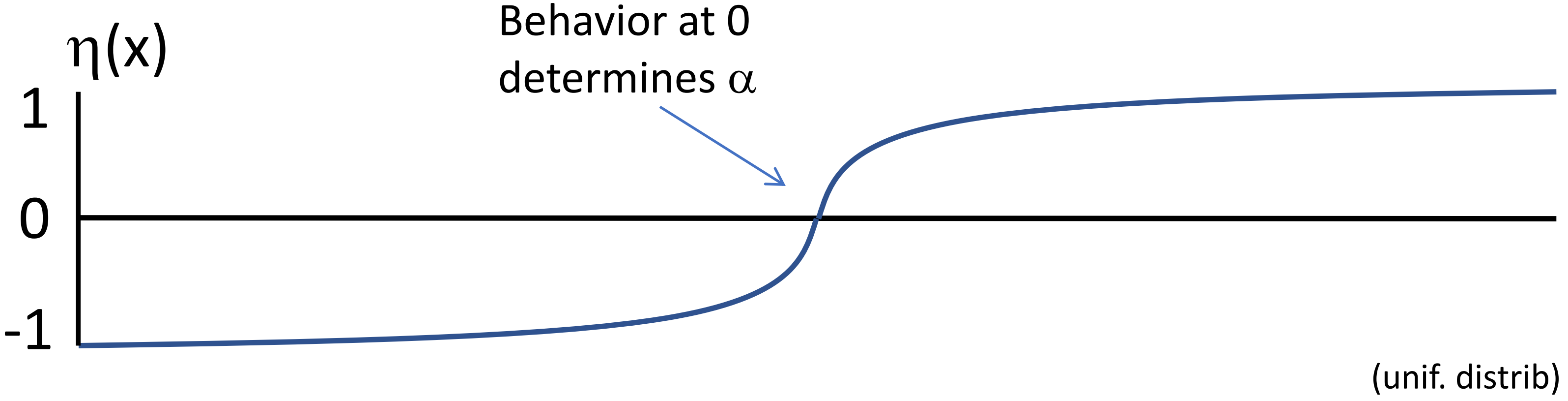
Example:
Thresholds



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Passive OPT: $\tilde{\Theta}\left(\frac{d}{\epsilon^{2-\alpha}}\right)$. (Massart & Nédélec, 2006)

Active OPT: (roughly)

$$\begin{cases} \frac{d}{\epsilon^{2-2\alpha}} & \text{if } 0 < \alpha \leq 1/2 \\ \min\left\{\frac{d}{\epsilon^{2-2\alpha}}\left(\frac{\mathfrak{s}}{d}\right)^{2\alpha-1}, \frac{d}{\epsilon}\right\} & \text{if } 1/2 < \alpha < 1 \end{cases}.$$

(Hanneke & Yang, 2015)

$$\sim \begin{cases} \frac{1}{\varepsilon^{2-2\alpha}}, & \text{if } \mathfrak{s} < \infty \\ \frac{1}{\varepsilon}, & \text{if } \mathfrak{s} = \infty \end{cases}.$$

Active Opt $\ll$ **Passive Opt.**
(always)

Conclusions

- Many proposals for going beyond Disagreement-based Active Learning
- Each exhibits improvements in certain cases
- We still don't know the **optimal agnostic active learning algorithm**

$$d\frac{\beta^2}{\epsilon^2} + \mathfrak{s}_{\epsilon/d}\log(\frac{1}{\epsilon})$$

Questions?

Further reading:

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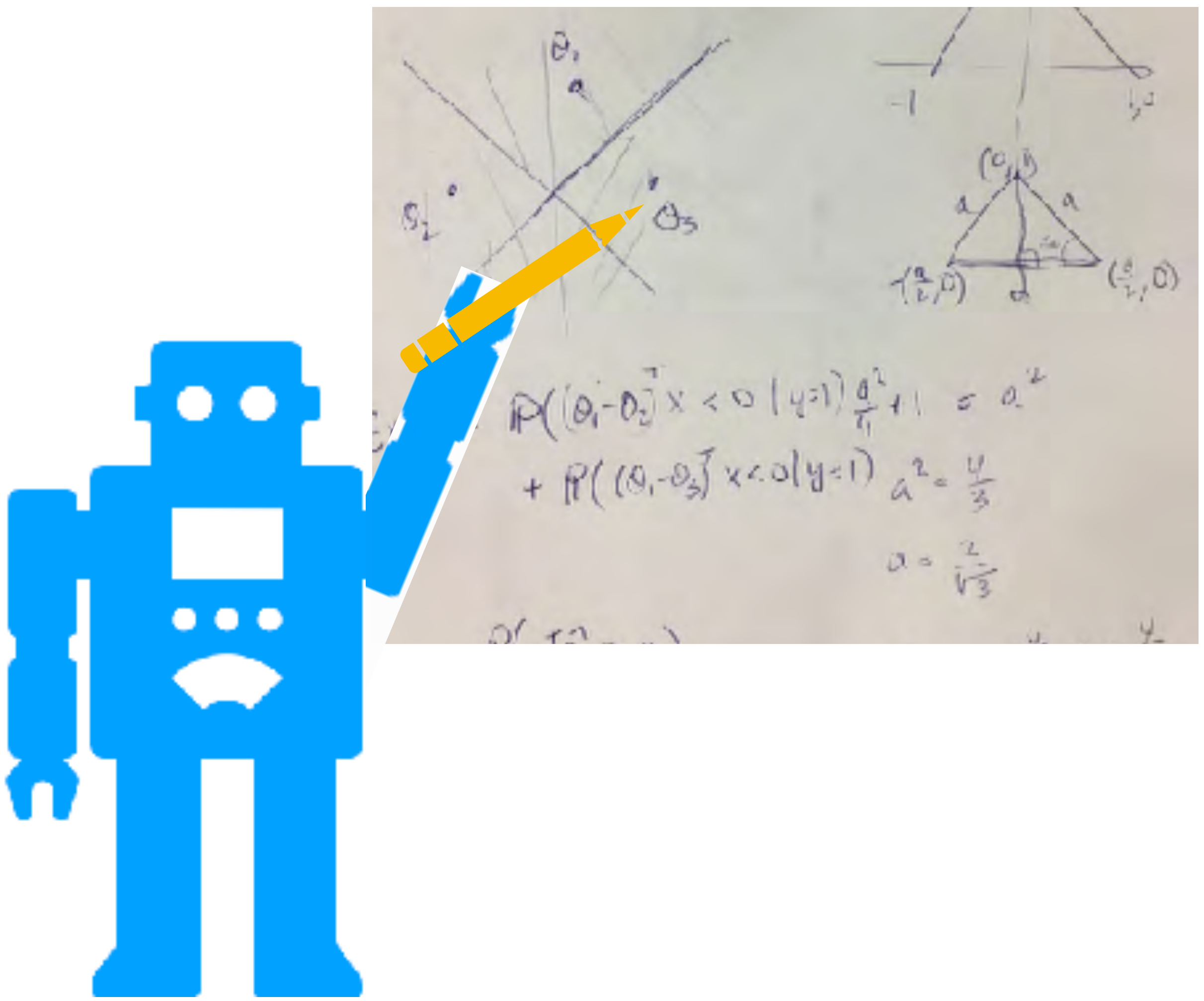
Active Learning from Theory to Practice



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ICML | 2019

Thirty-sixth International Conference on Machine Learning

Tutorial Outline

Part 1: Introduction to Active Learning (Rob)

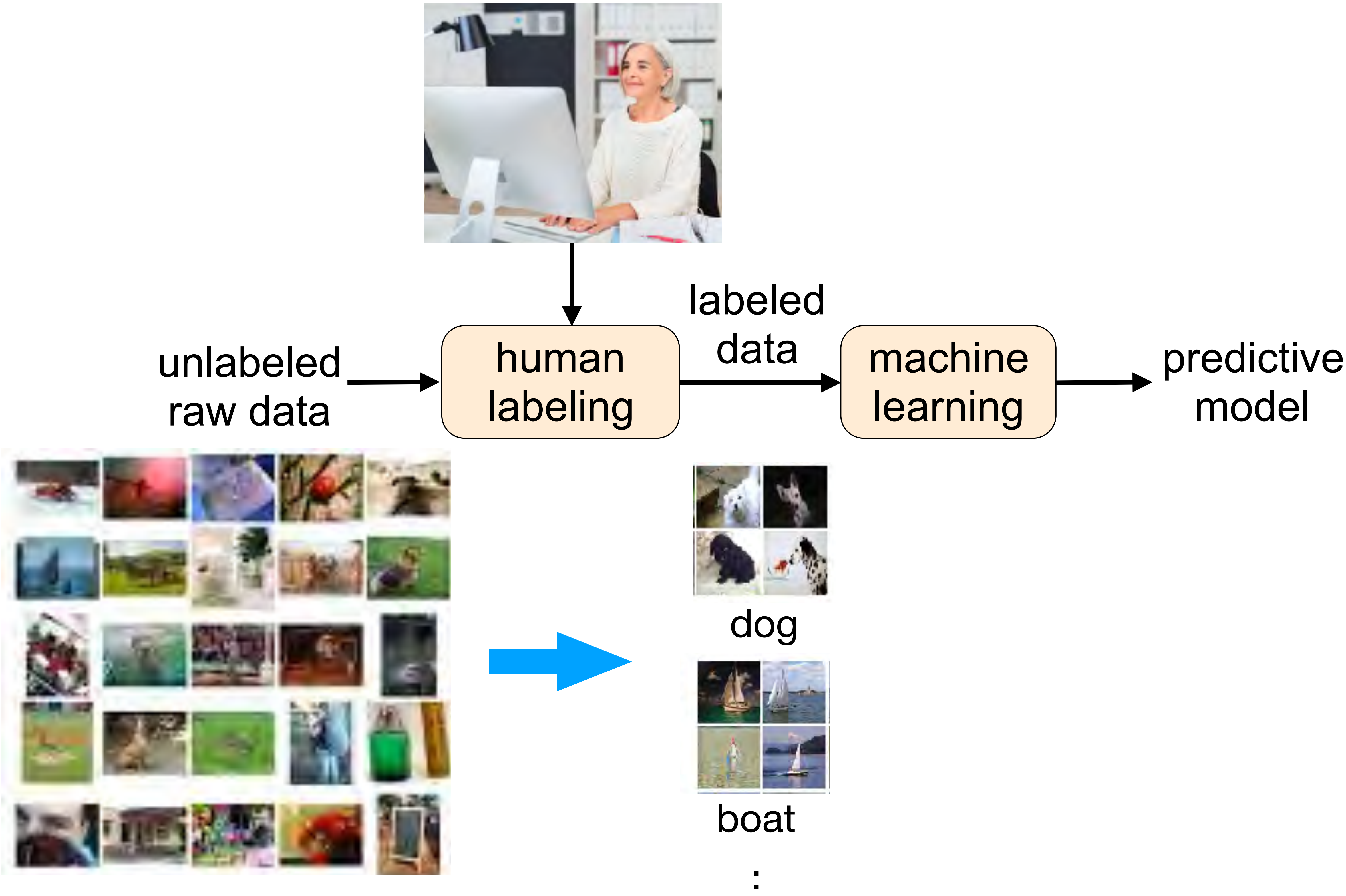
Part 2: Theory of Active Learning (Steve)

Part 3: Advanced Topics and Open Problems (Steve)

Part 4: Nonparametric Active Learning (Rob)

slides: http://nowak.ece.wisc.edu/ActiveML.html

Conventional (Passive) Machine Learning



ALL SYSTEMS GO ?

theguardian

Computers now better than humans at recognising and sorting images

millions of labeled images
1000's of human hours

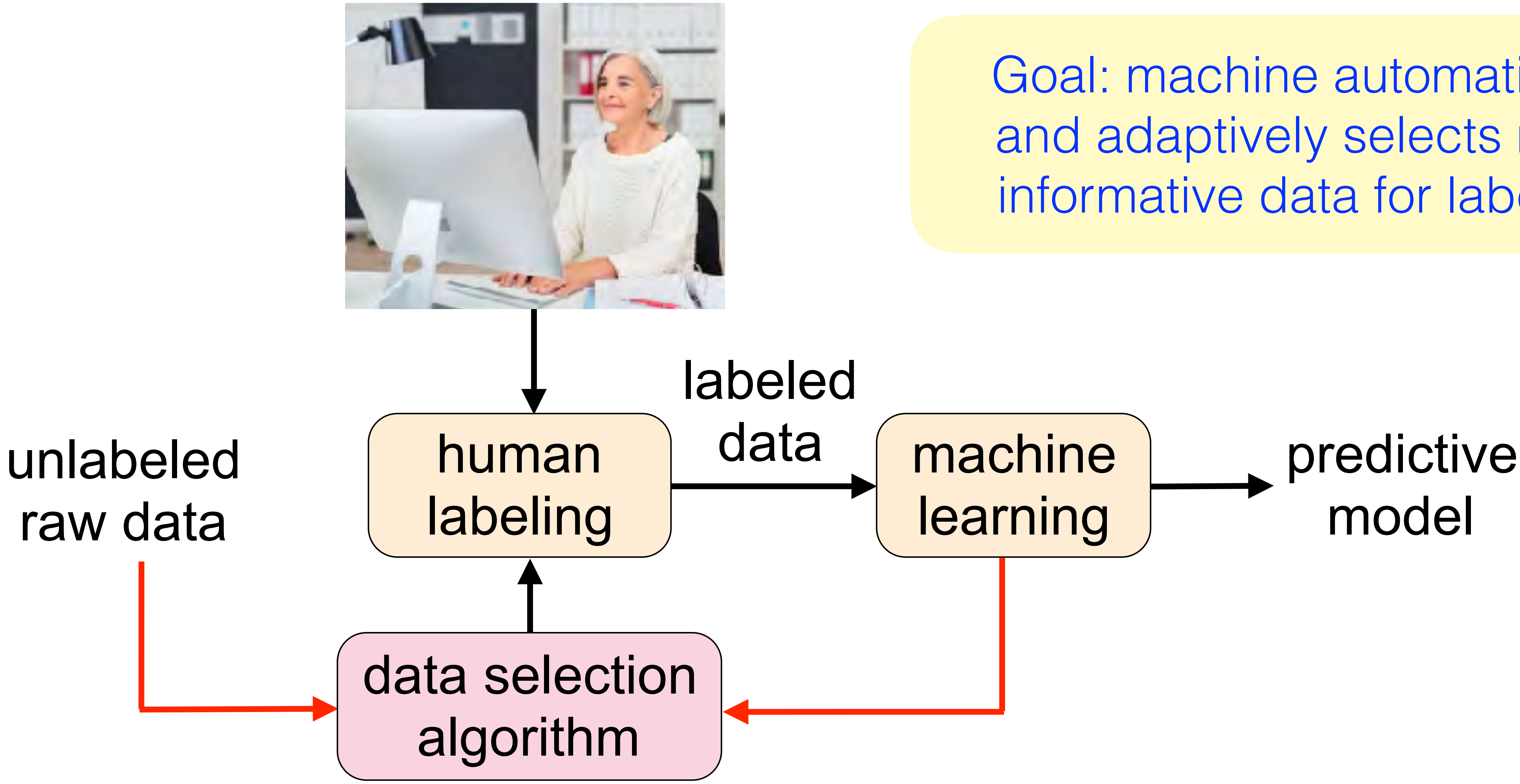
QUARTZ

Google says its new AI-powered translation tool scores nearly identically to human translators

trained on more texts than a human could read in a lifetime

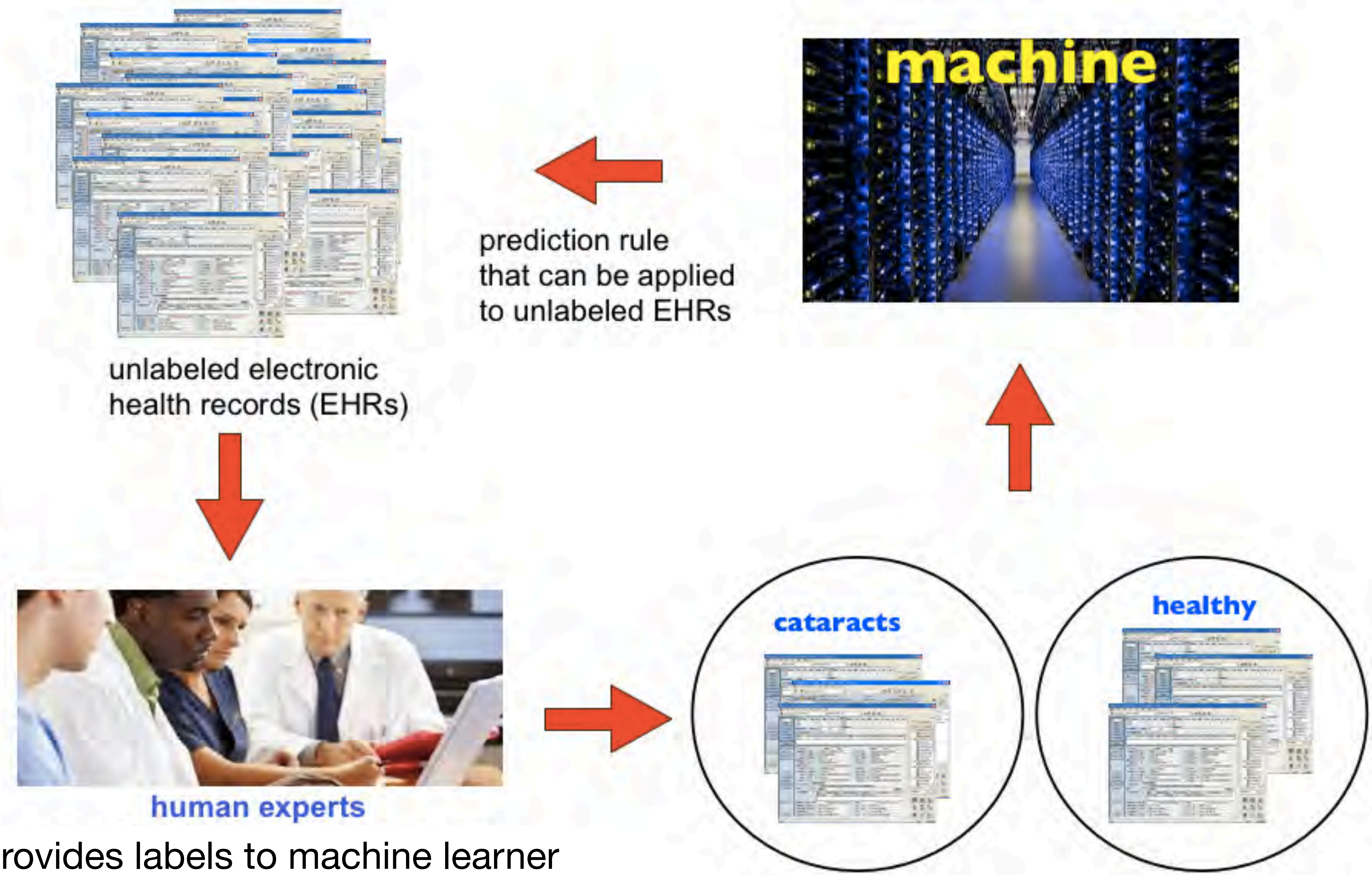
Can we train machines with less labeled data and less human supervision?

Active Machine Learning



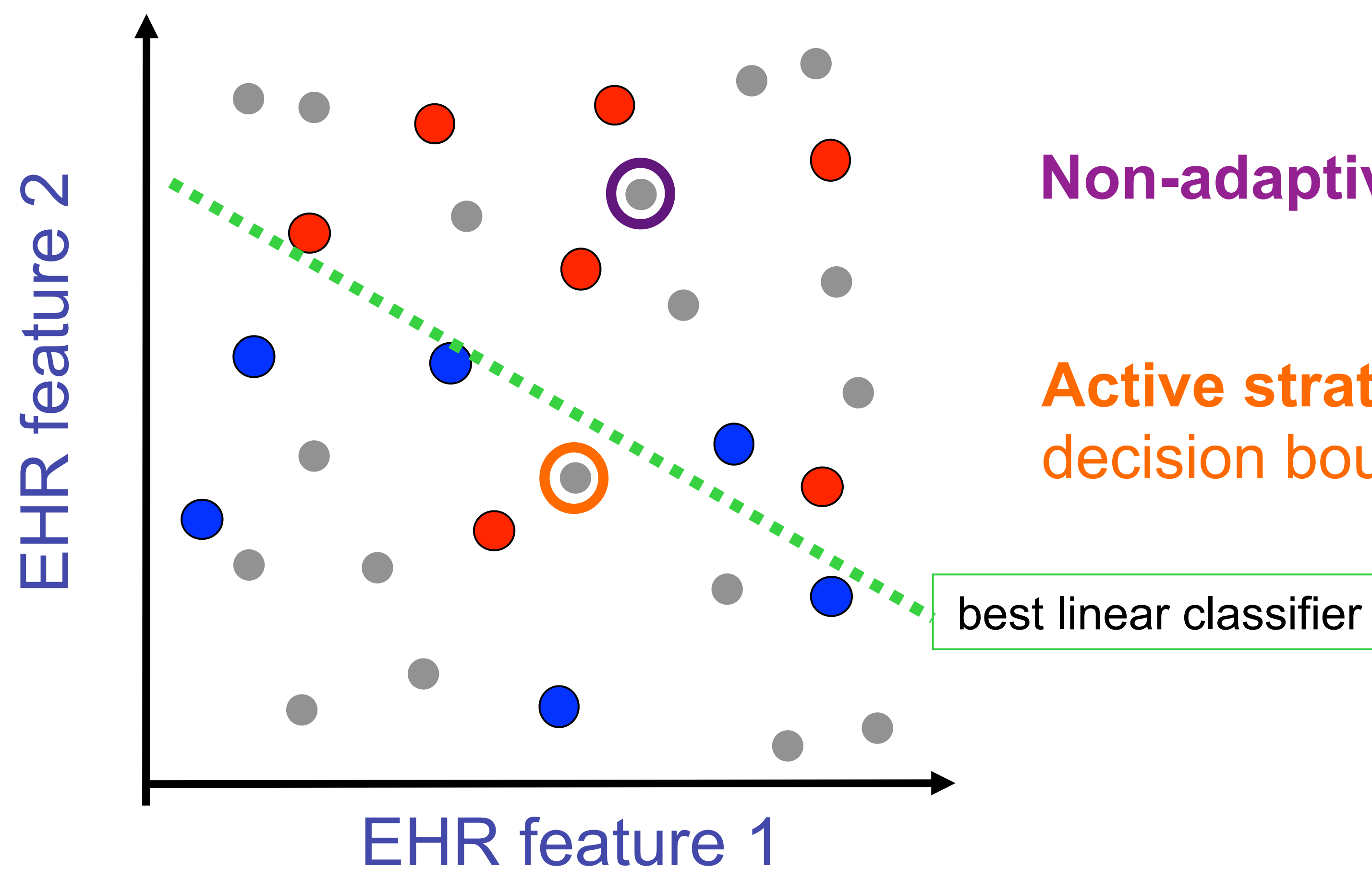
Goal: machine automatically and adaptively selects most informative data for labeling

Motivating Application



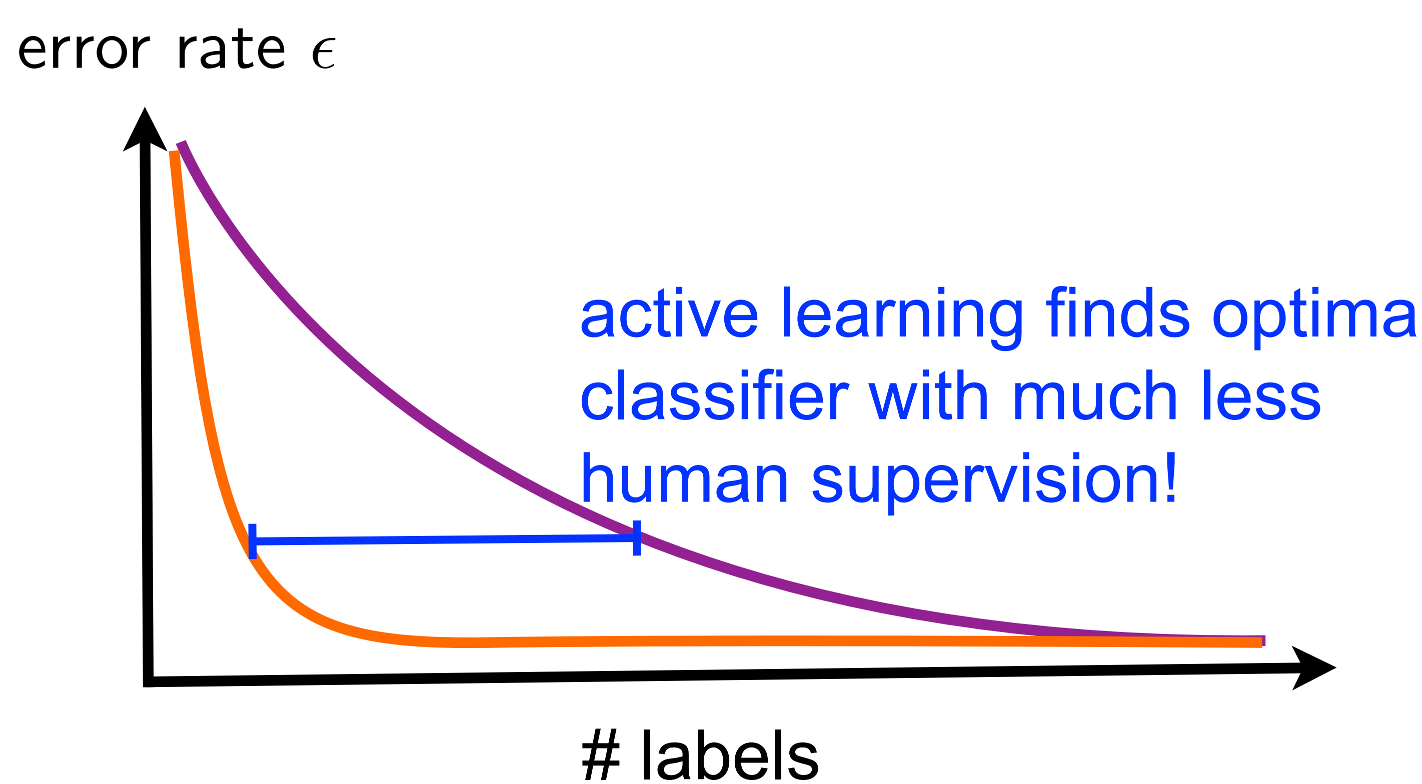
provides labels to machine learner
(several minutes / EHR)

Active Learning

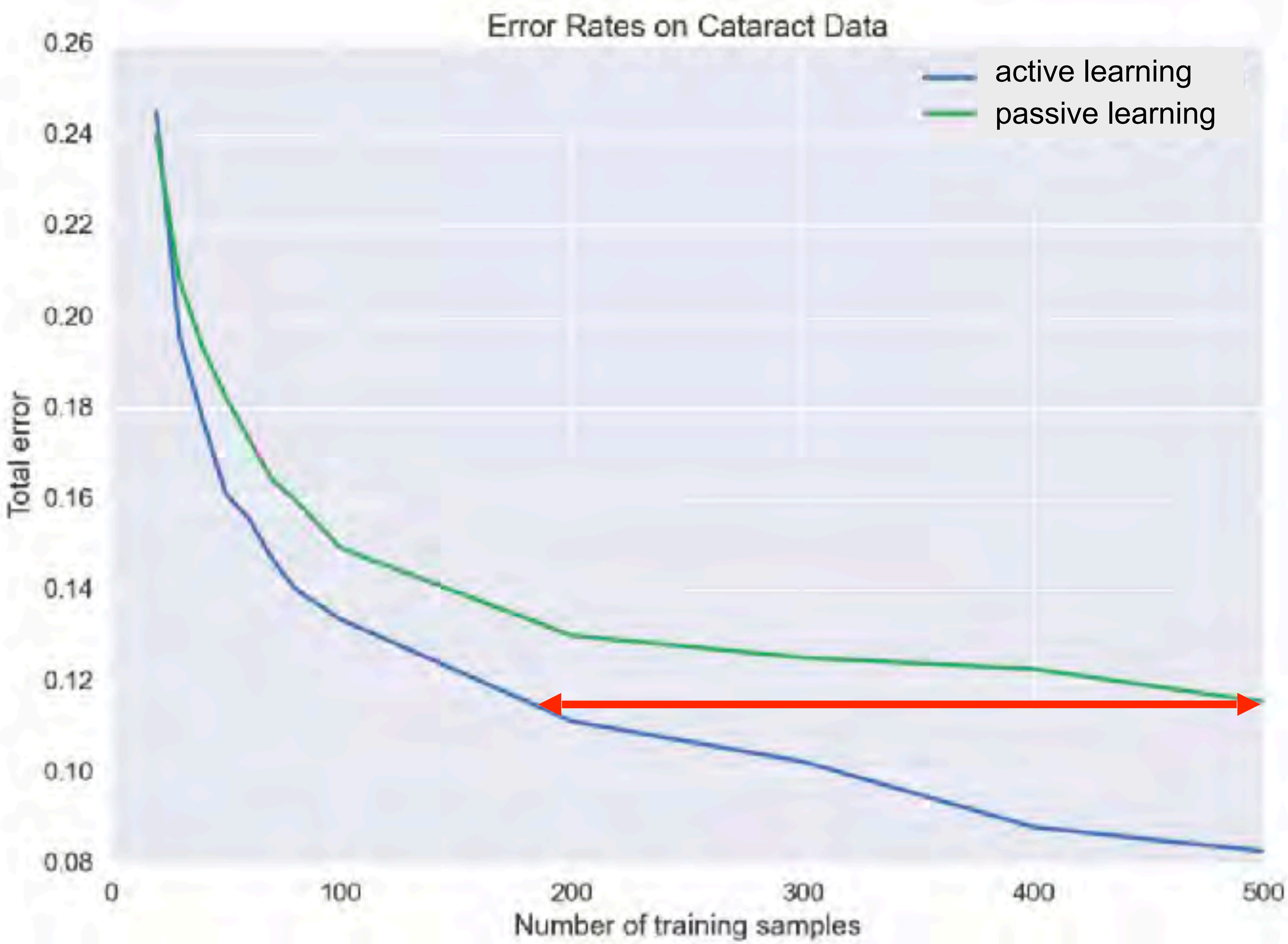


Non-adaptive strategy: Label a random sample

Active strategy: Label a sample near best decision boundary based on labels seen so far



Active Logistic Regression



11000 patient records
8000 positive
3000 negative

6182 Numerical Features
icd9 codes
lab tests
patient data

Classification task:
cataracts or healthy

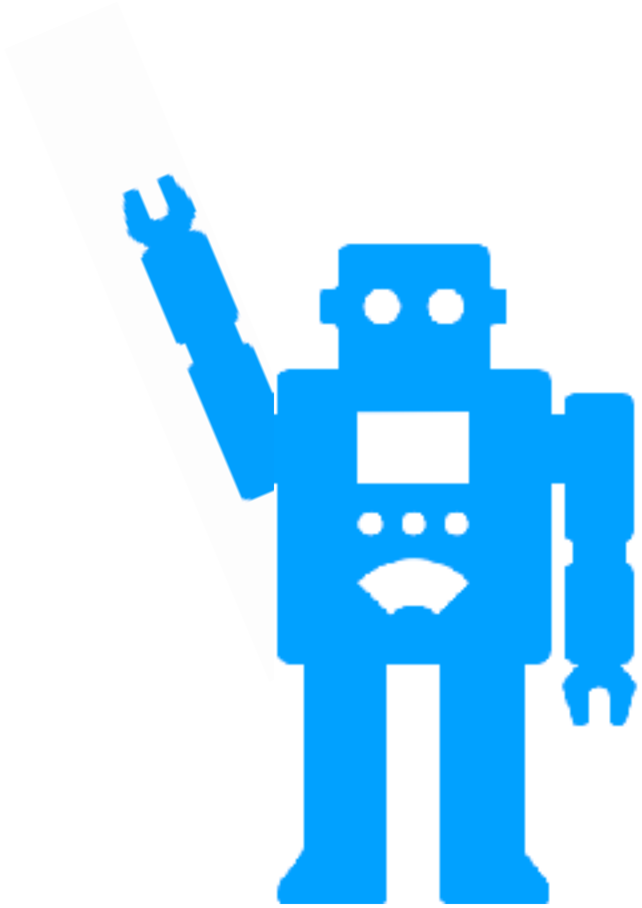
less than half as many labeled examples needed by active learning


nextml.org
NEXT
ASK BETTER QUESTIONS.
GET BETTER RESULTS.
FASTER. AUTOMATED.
GitHub
Paper
Docs
Blog
Team
Data

Active learning to optimize crowdsourcing and rating in New Yorker Cartoon Caption Contest

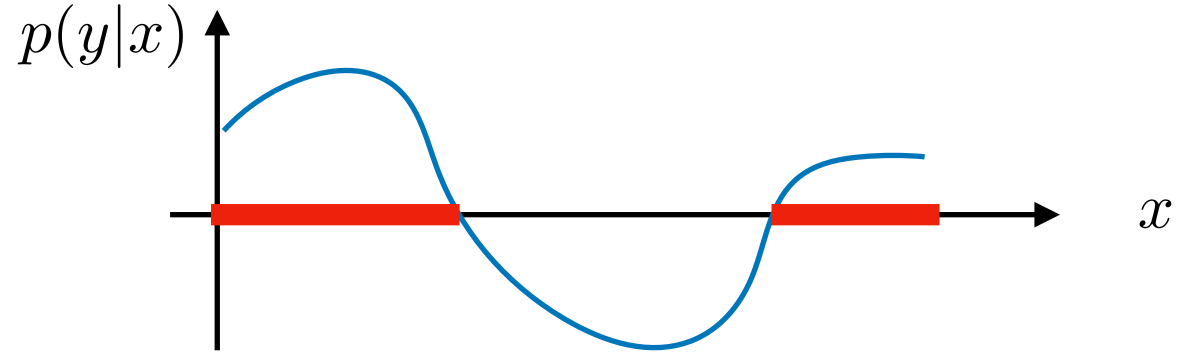
Actively learning user’s beer preferences

Principles of Active Learning

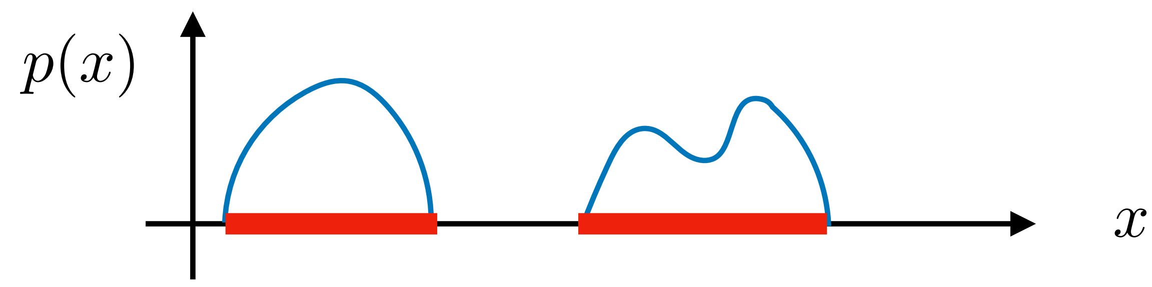


What and Where Information

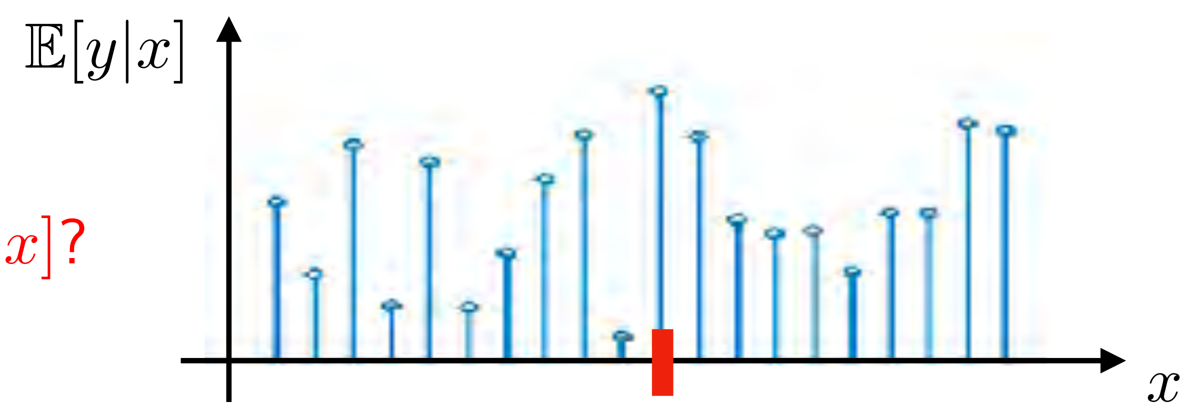
Density estimation: What is $p(y|x)$?
Classification: Where is $p(y|x) > 0$?



Density estimation: What is $p(x)$?
Clustering: Where is $p(x) > \epsilon$?



Function estimation: What is $\mathbb{E}[y|x]$?
Bandit optimization: Where is $\max_x \mathbb{E}[y|x]$?



Active learning is more efficient than passive learning for localized “where” information

Meta-Algorithm for Active Learning

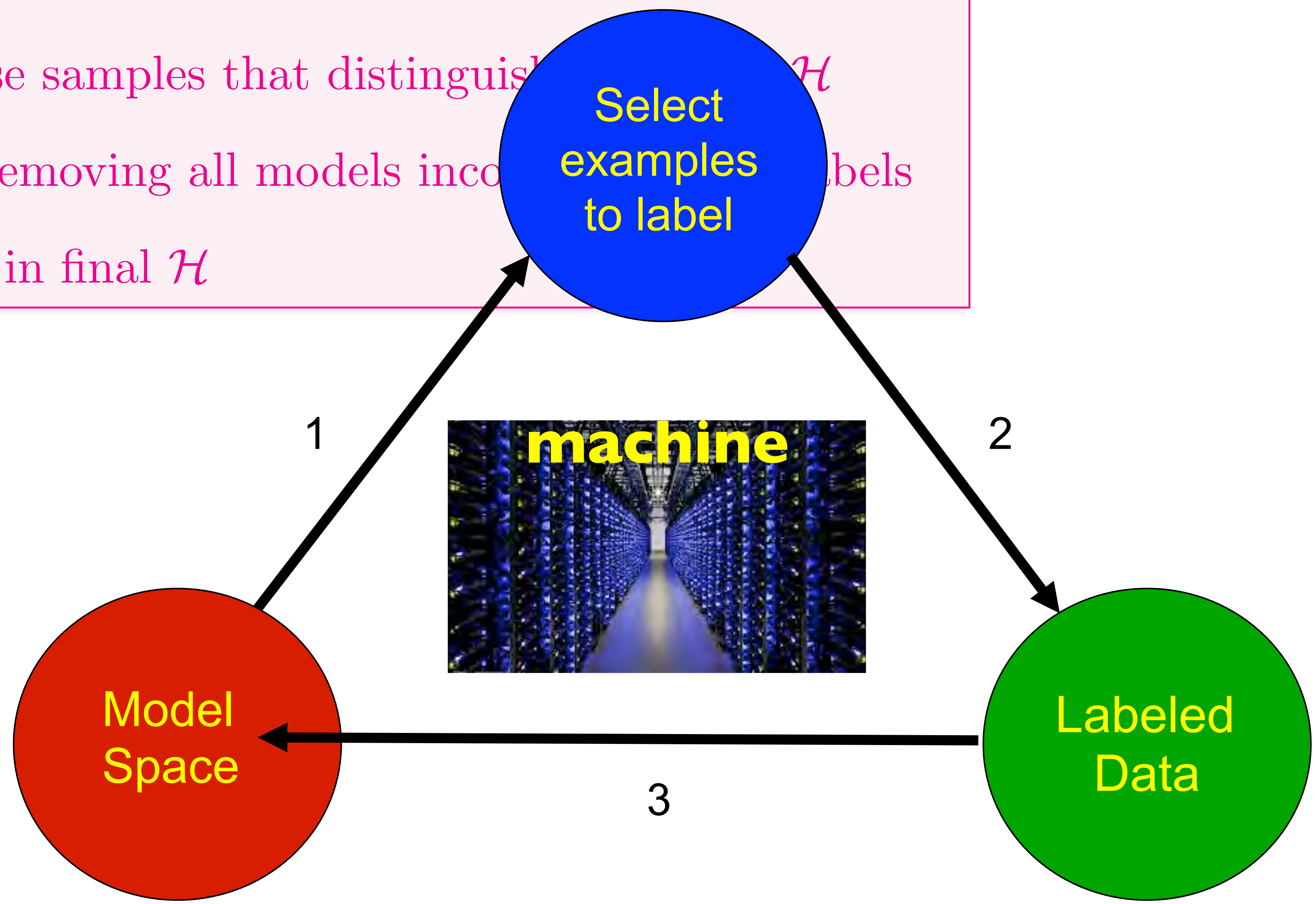
Version-Space (VS) Active Learning

initialize VS: $\mathcal{H}$ = all models/hypotheses

while (*stopping-criterion*) not met

1. **sample** at random from available dataset
2. **label** only those samples that distinguis $\mathcal{H}$
3. **reduce** $\mathcal{H}$ by removing all models inco bels

output: best model in final $\mathcal{H}$



Learning a 1-D Classifier

binary search quickly finds **decision boundary**

passive : err $\sim \quad n^{-1}$

active : err $\sim \quad 2^{-n}$

Vapnik-Chervonenkis (VC) Theory

Given training data $\{(x_j, y_j)\}_{j=1}^n$, learn a function f to predict y from x

Consider a possibly infinite set of hypotheses $\mathcal{F}$ with *finite VC dimension* d and for each $f \in \mathcal{F}$ define the risk (error rate):

$$R(f) \; := \; \mathbb{P}(f(x) \neq y)$$

error rate on training data:

$$\widehat{R}(f) \; = \; \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\Big(f(x_i) \neq y_i\Big)$$

"empirical risk"

VC bound:

$$\sup_{f \in \mathcal{F}} |R(f) - \widehat{R}(f)| \; \leq \; 6 \sqrt{\frac{d \log(n/\delta)}{n}}$$

w.p. $\geq \; 1 - \delta$

Empirical Risk Minimization (ERM)

Goal: select hypothesis with true error rate within $\epsilon > 0$ of $\min_{f \in \mathcal{F}} R(f)$

$$f^* = \arg\min_{f \in \mathcal{F}} R(f)$$ true risk minimizer

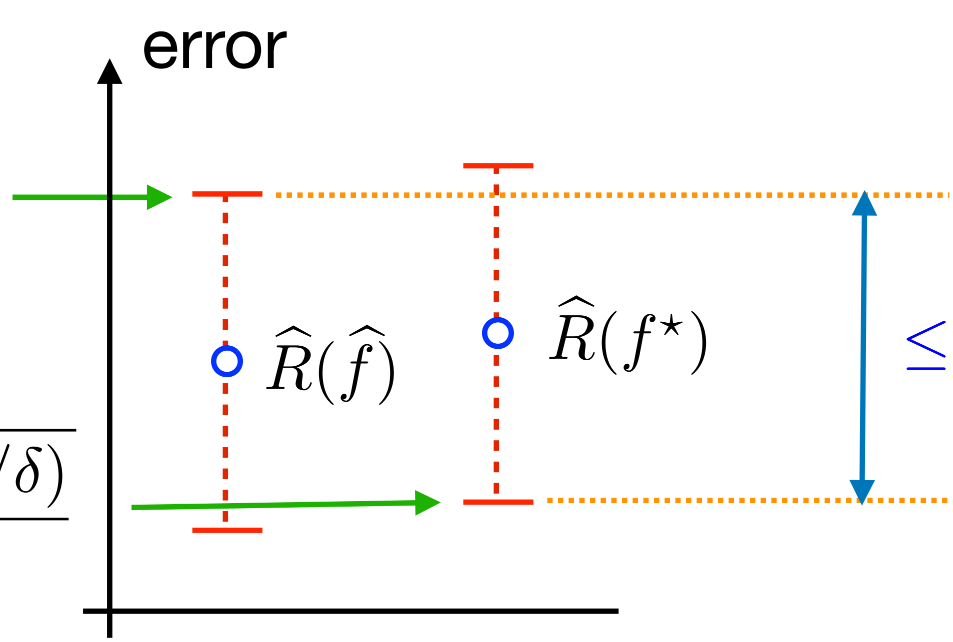
$$\widehat{f} = \arg\min_{f \in \mathcal{F}} \widehat{R}(f)$$ empirical risk minimizer

$\widehat{f}$ minimizes empirical risk:

$$\widehat{R}(\widehat{f}) \leq \widehat{R}(f^*)$$

$$R(\widehat{f}) \leq \widehat{R}(\widehat{f}) + 6\sqrt{\frac{d\log(n/\delta)}{n}}$$

$$R(f^*) \geq \widehat{R}(f^*) - 6\sqrt{\frac{d\log(n/\delta)}{n}}$$



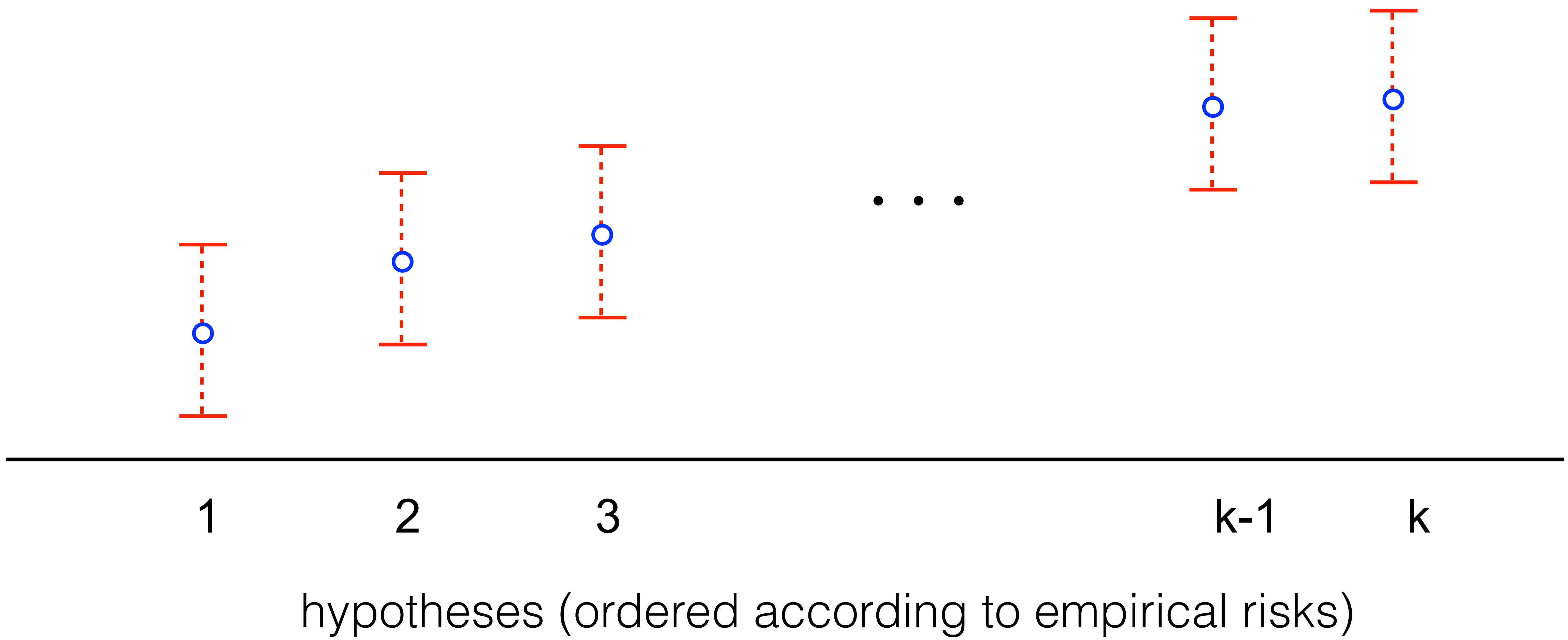
$$\leq \quad 12\sqrt{\frac{d\log(n/\delta)}{n}}$$

sufficient number of training examples:

$$12\sqrt{\frac{d\log(n/\delta)}{n}} \leq \epsilon$$

$$n = \widetilde{O}\Big(\frac{d\log(1/\delta)}{\epsilon^2}\Big)$$

Empirical Risks and Confidence Intervals



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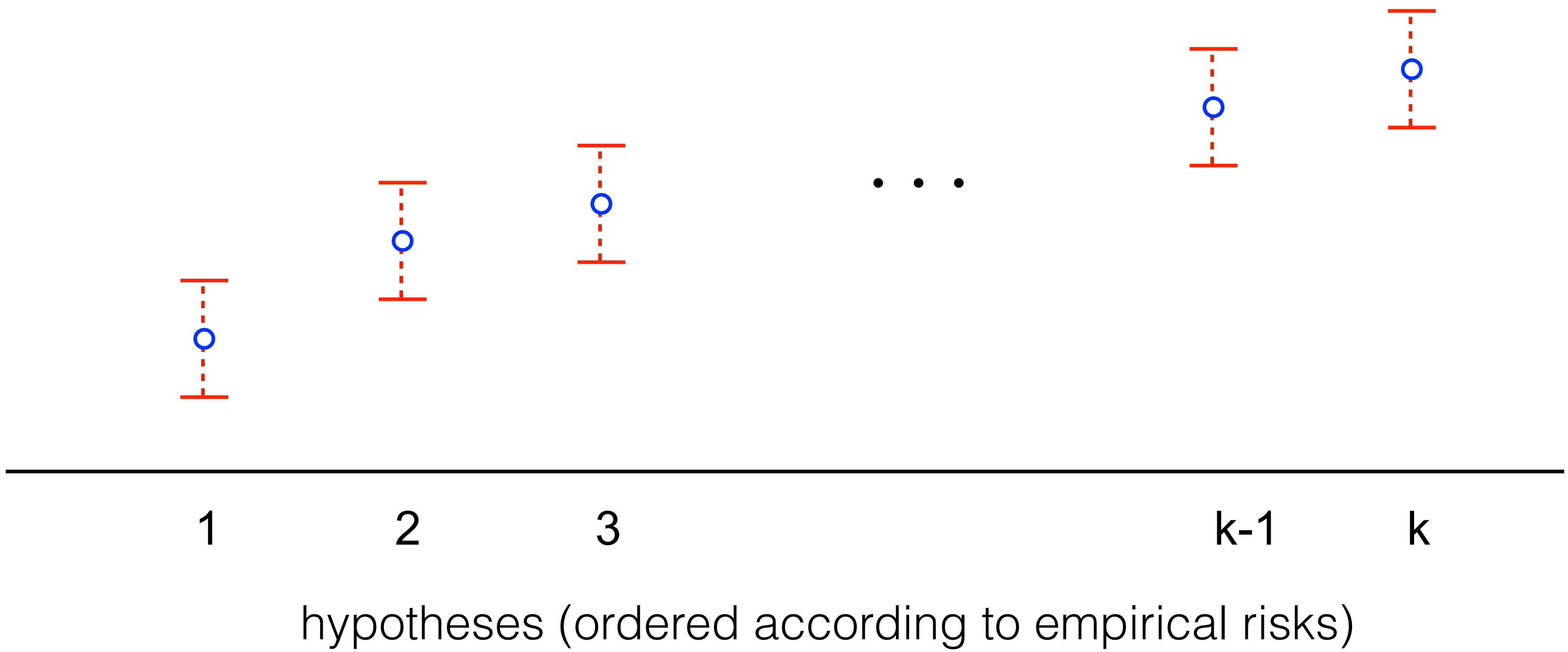
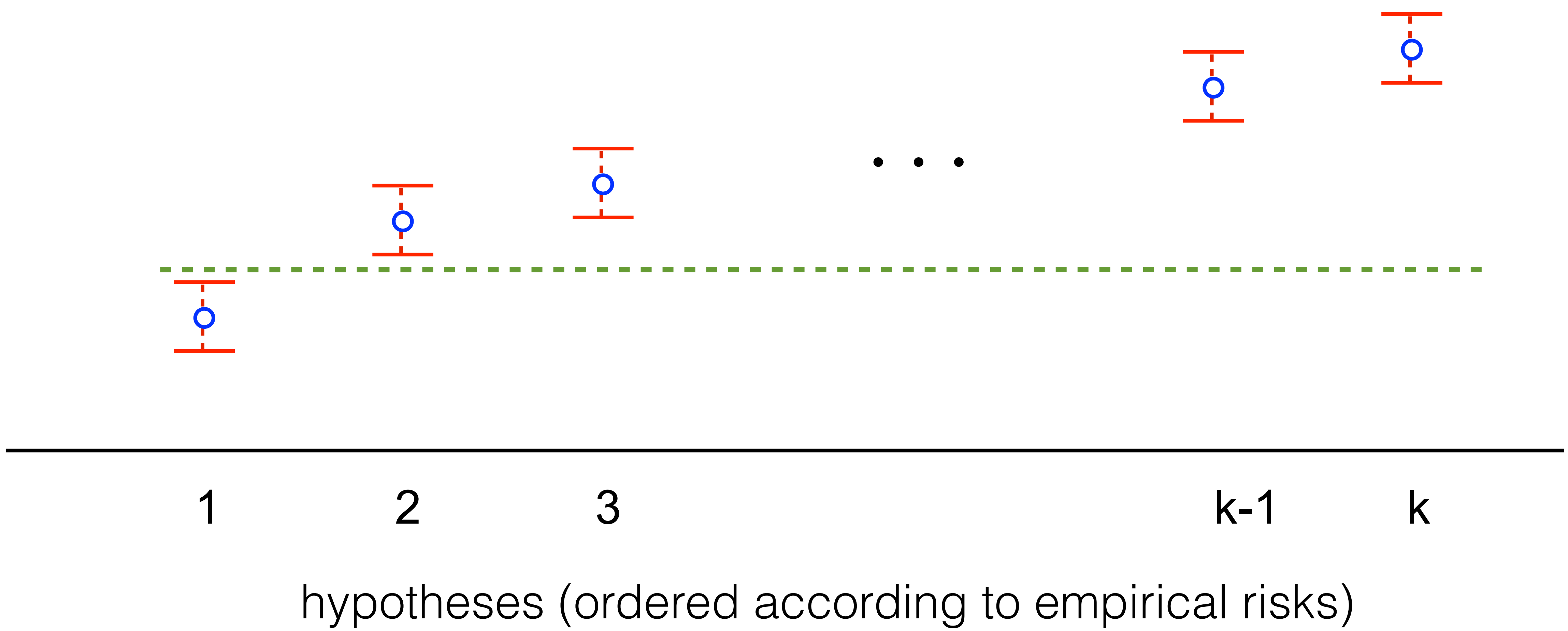


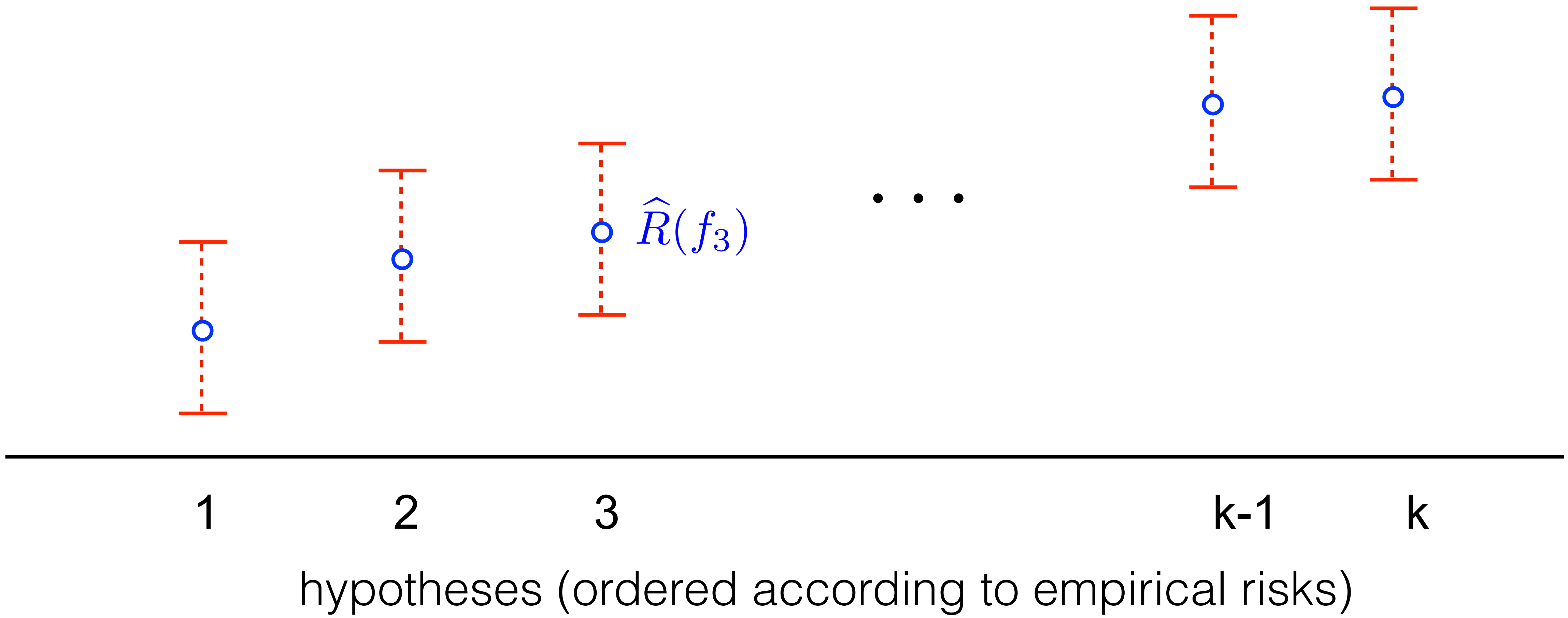
more training data $\Rightarrow$ smaller confidence intervals

Empirical Risks and Confidence Intervals



more training data ⇒ smaller confidence intervals

ERM is Wasting Labeled Examples



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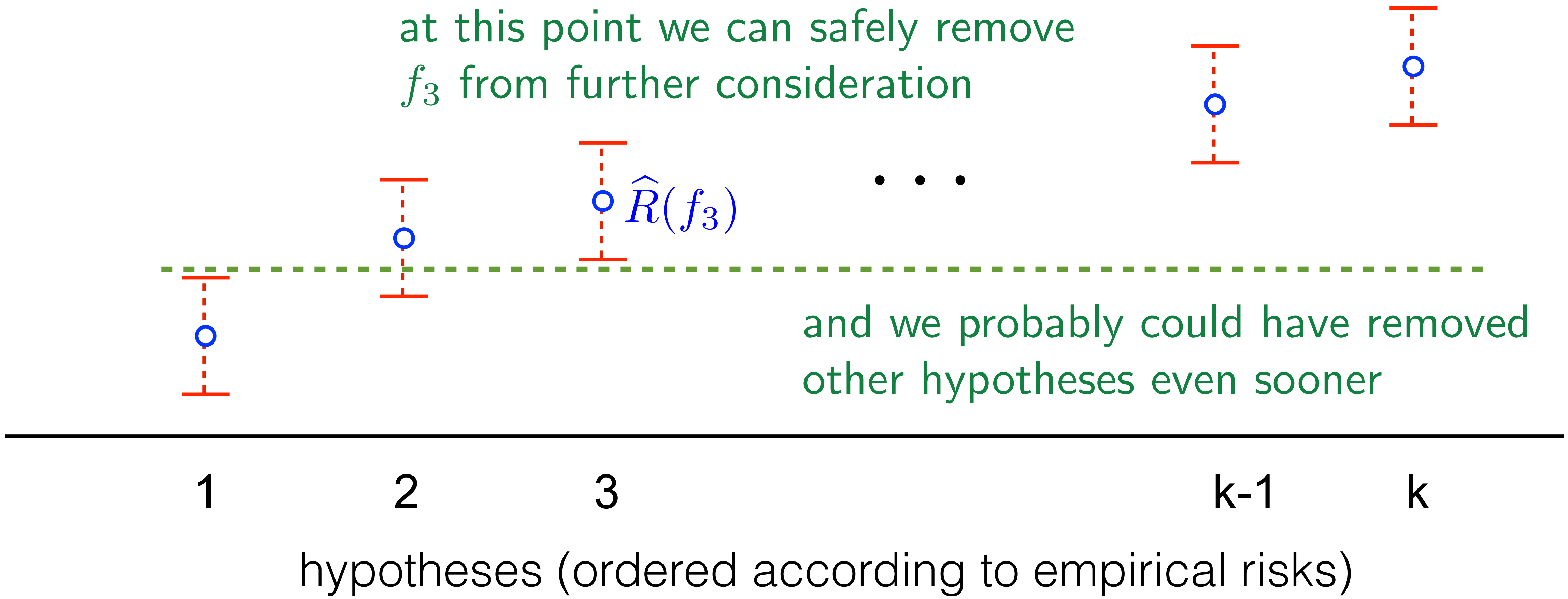
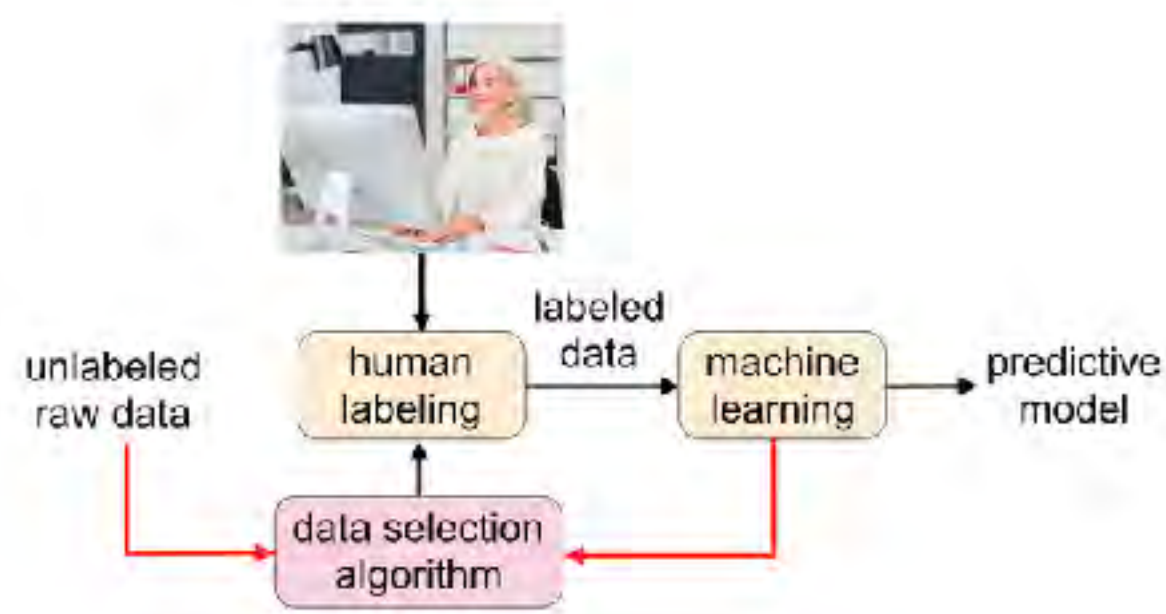
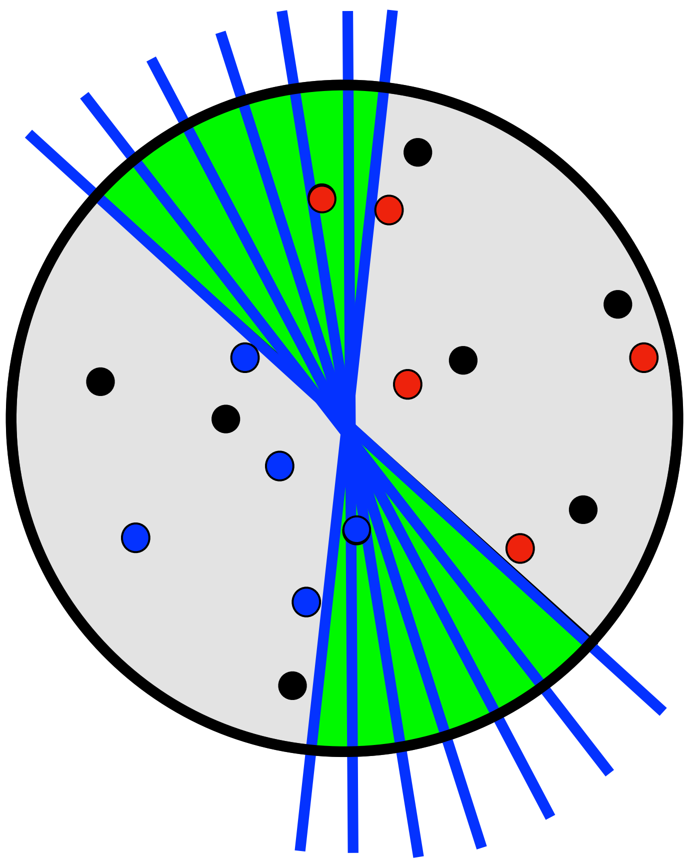


only require labels for examples that hypotheses 1 and 2 label differently (i.e., examples where they *disagree*)



Disagreement-Based Active Learning

consider points uniform on unit ball and
linear classifiers passing through origin



only label points in the
region of disagreement $\mathfrak{D}$

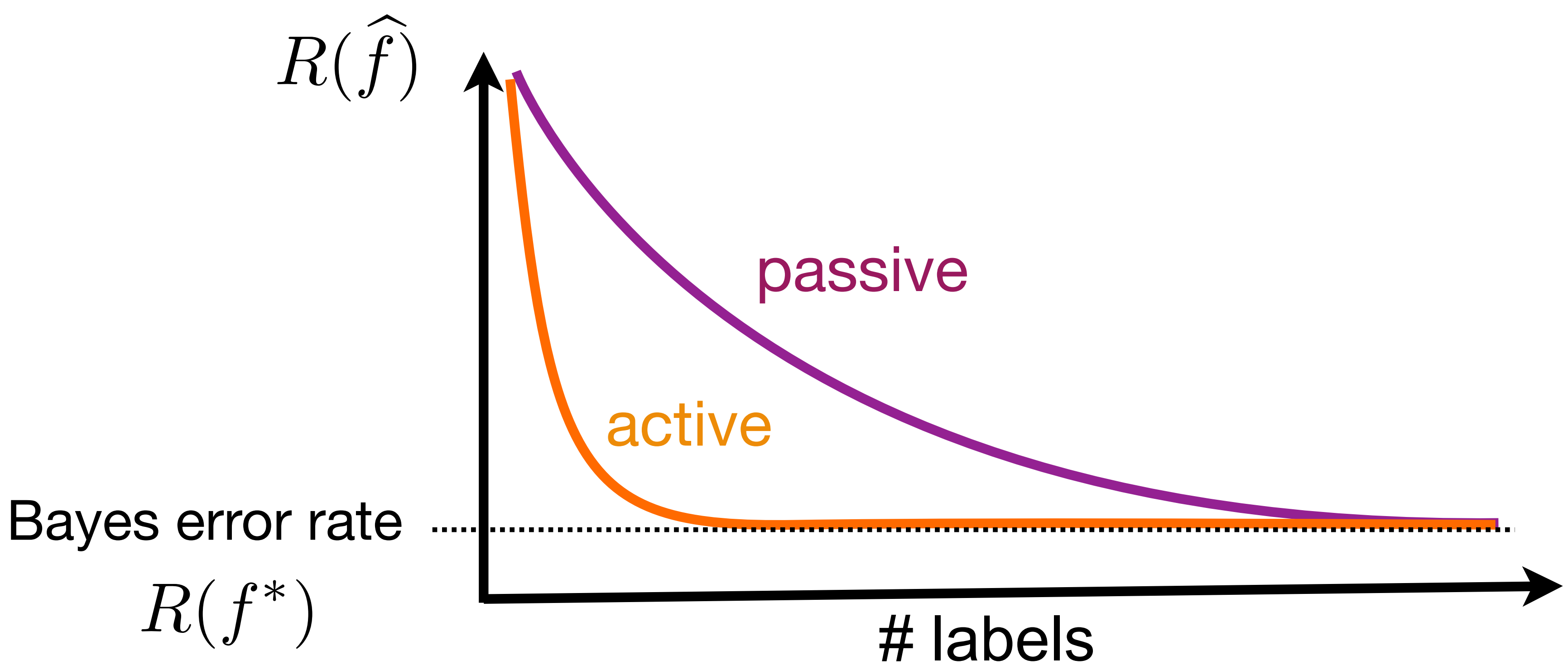
Active Binary Classification

Assuming optimal Bayes classifer f^* in VC class with dimension d and "nice" distributions (e.g., bounded label noise)

$$\epsilon = R(\widehat{f}) - R(f^*)$$

passive $\quad \epsilon \sim \frac{d}{n} \quad$ parametric rate

active $\quad \epsilon \sim \exp\left(-c\,\frac{n}{d}\right) \quad$ exponential speed-up



Tutorial Outline

Part 1: Introduction to Active Learning (Rob)

Part 2: Theory of Active Learning (Steve)

Part 3: Advanced Topics and Open Problems (Steve)

Part 4: Nonparametric Active Learning (Rob)

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Recommended Reading (Foundations of Active Learning)

Settles, Burr. "Active learning." *Synthesis Lectures on Artificial Intelligence and Machine Learning* 6.1 (2012): 1-114.

Dasgupta, Sanjoy. "Two faces of active learning." *Theoretical computer science* 412.19 (2011): 1767-1781.

Cohn, David, Les Atlas, and Richard Ladner. "Improving generalization with active learning." *Machine learning* 15.2 (1994): 201-221.

Castro, Rui M., and Robert D. Nowak. "Minimax bounds for active learning." *IEEE Transactions on Information Theory* 54, no. 5 (2008): 2339-2353.

Zhu, Xiaojin, John Lafferty, and Zoubin Ghahramani. "Combining active learning and semi-supervised learning using gaussian fields and harmonic functions." *ICML 2003 workshop*. Vol. 3. 2003.

Dasgupta, Sanjoy, Daniel J. Hsu, and Claire Monteleoni. "A general agnostic active learning algorithm." *Advances in neural information processing systems*. 2008.

Balcan, Maria-Florina, Alina Beygelzimer, and John Langford. "Agnostic active learning." *Journal of Computer and System Sciences* 75.1 (2009): 78-89.

Nowak, Robert D. "The geometry of generalized binary search." *IEEE Transactions on Information Theory* 57, no. 12 (2011): 7893-7906.

Hanneke, Steve. "Theory of active learning." *Foundations and Trends in Machine Learning* 7, no. 2-3 (2014).

Part 2: Theory of Active Learning
General Case

- Disagreement-Based Agnostic Active Learning
- Disagreement Coefficient
- Sample Complexity Bounds

Tutorial on Active Learning: Theory to Practice

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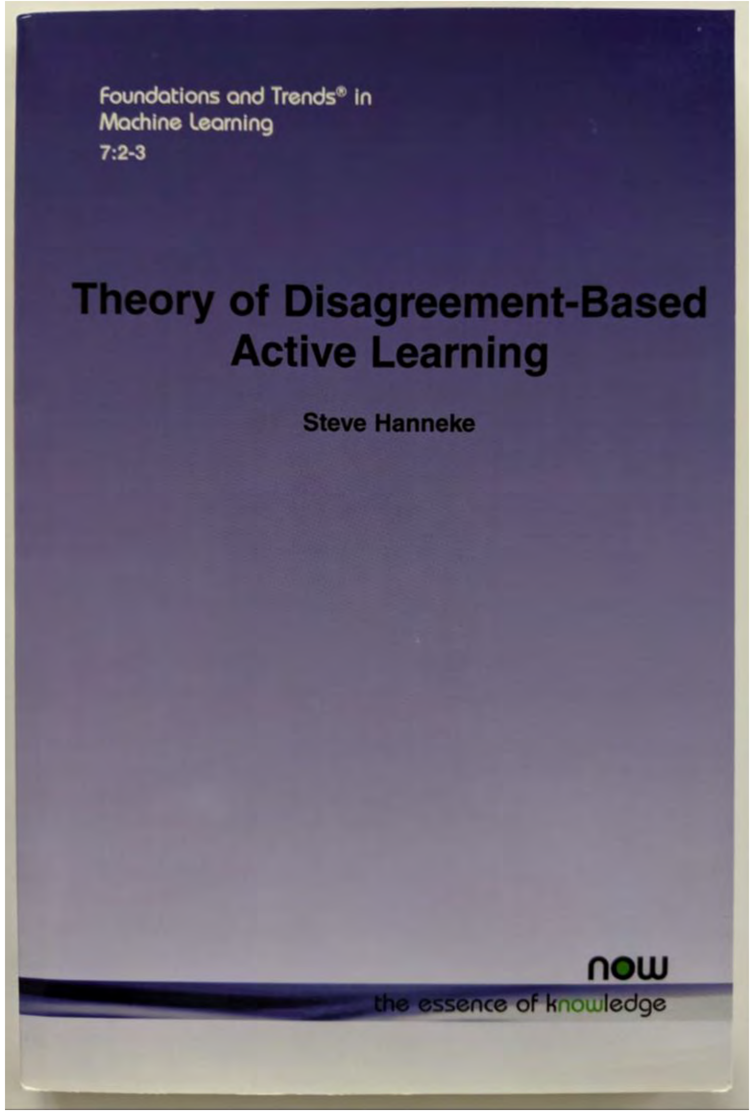
ICML | 2019
Thirty-sixth International Conference on Machine Learning

Agnostic Active Learning



Uniform Bernstein Inequality

Bernstein's inequality:

For m iid samples

$\forall f, f'$, w.p. $1-\delta$,

$$R(f) - R(f') \leq \hat{R}(f) - \hat{R}(f') + c\sqrt{\hat{P}(f \neq f')\frac{\log(1/\delta)}{m}} + \frac{\log(1/\delta)}{m}$$

Uniform Bernstein inequality:

VC dimension

w.p. $1-\delta$, $\forall f, f' \in \mathcal{H}$,

$$R(f) - R(f') \leq \hat{R}(f) - \hat{R}(f') + c\sqrt{\hat{P}(f \neq f')\frac{d\log(m/\delta)}{m}} + \frac{d\log(m/\delta)}{m}$$

Roughly:

$\forall f, f' \in \mathcal{H}$,

$$R(f) - R(f') \leq \hat{R}(f) - \hat{R}(f') + \sqrt{\hat{P}(f \neq f')\frac{d}{m}}$$

Agnostic Active Learning

Balcan, Beygelzimer, & Langford (2006)

Region of disagreement:

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
2. **label** points in $Q = \mathrm{DIS}(\mathcal{H}) \cap S$
3. **optimize** $\hat{f} \leftarrow \underset{f \in \mathcal{H}}{\mathrm{argmin}}\, \hat{R}_Q(f)$
4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$.

output final $\hat{f}$

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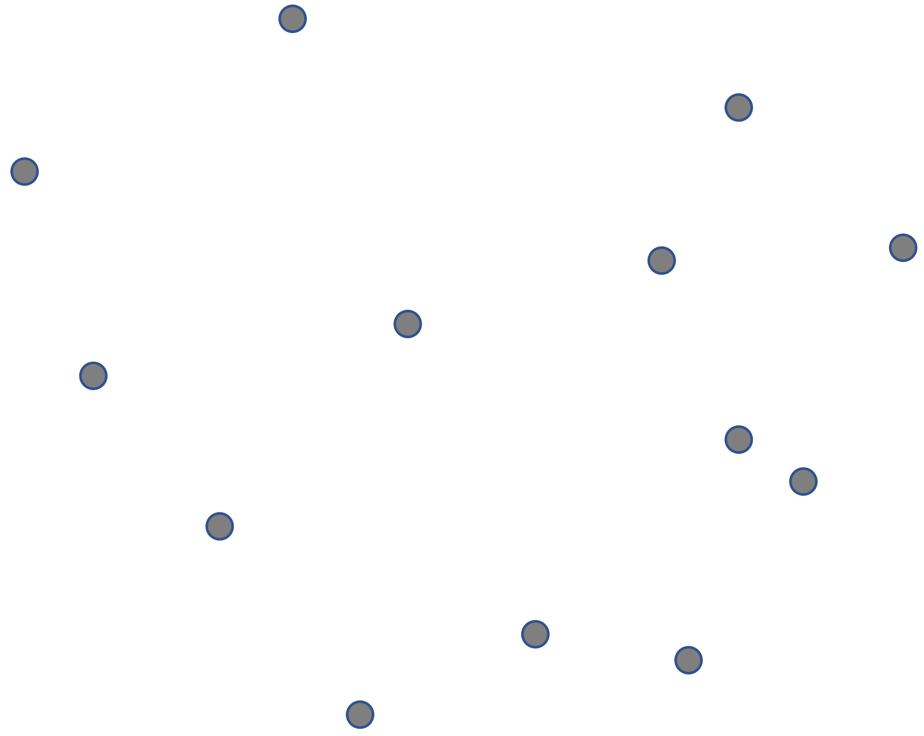

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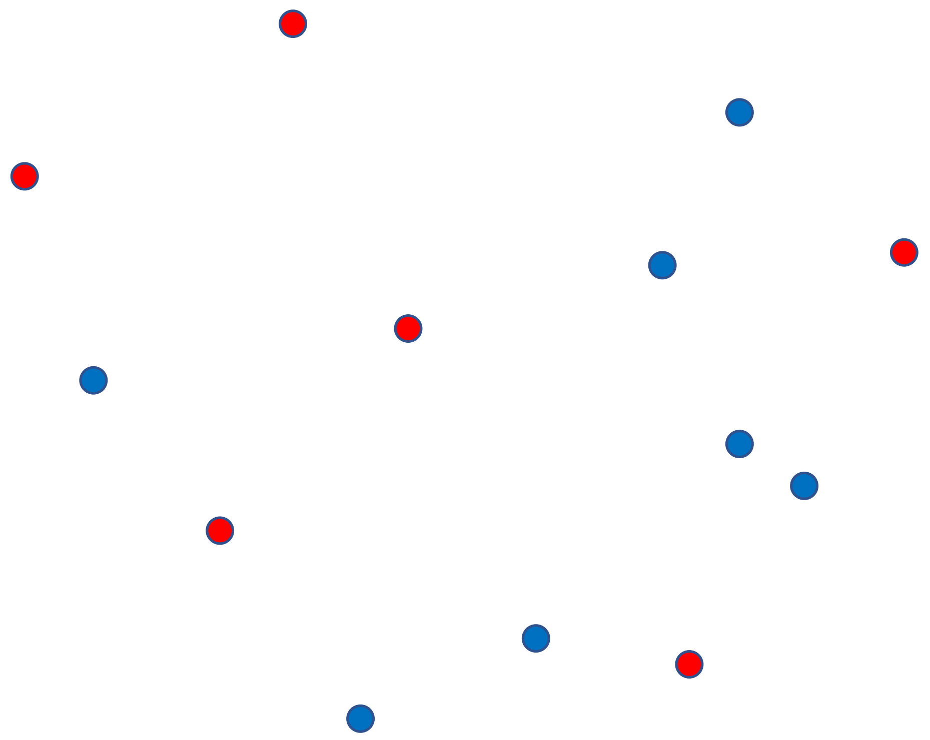

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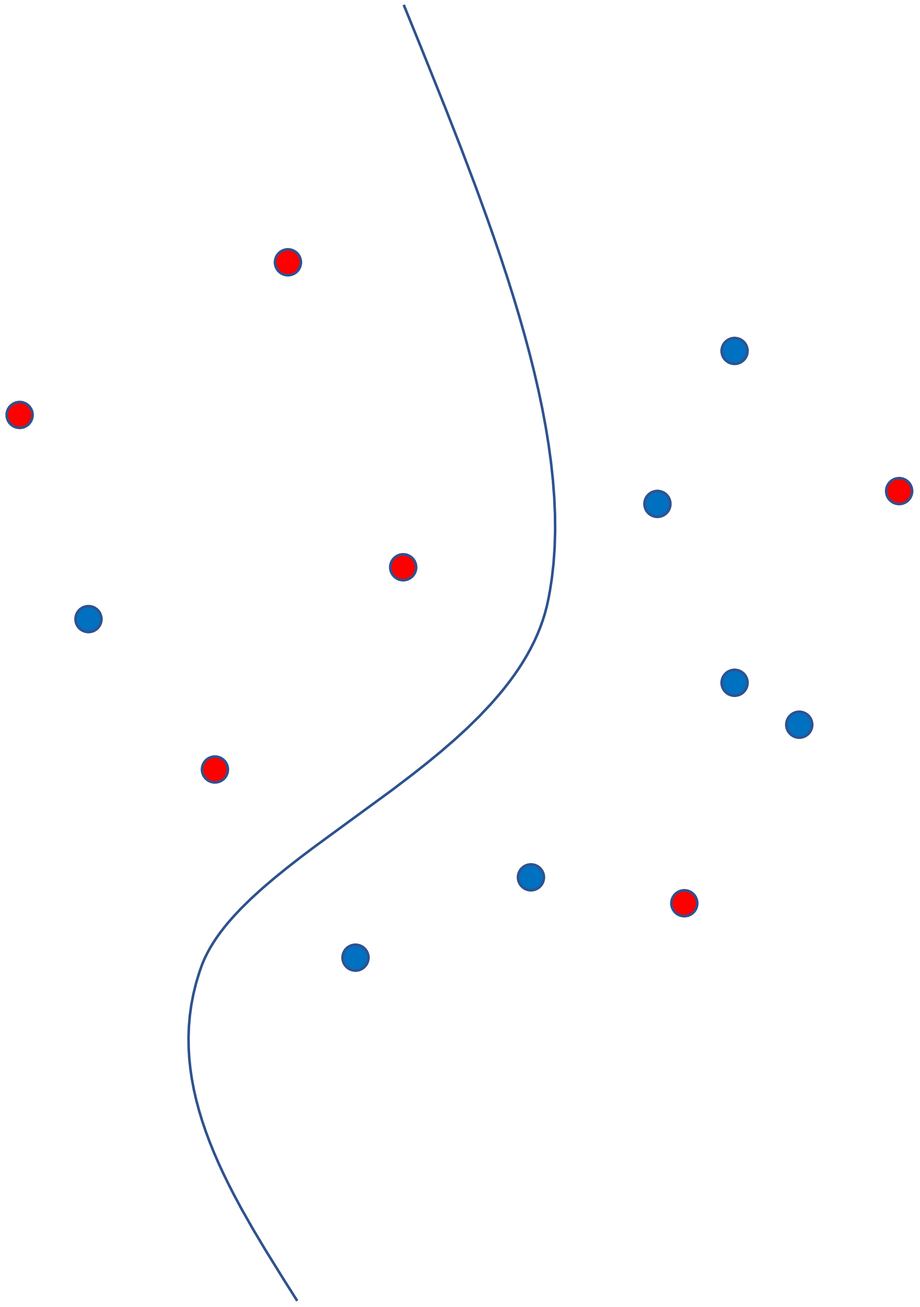

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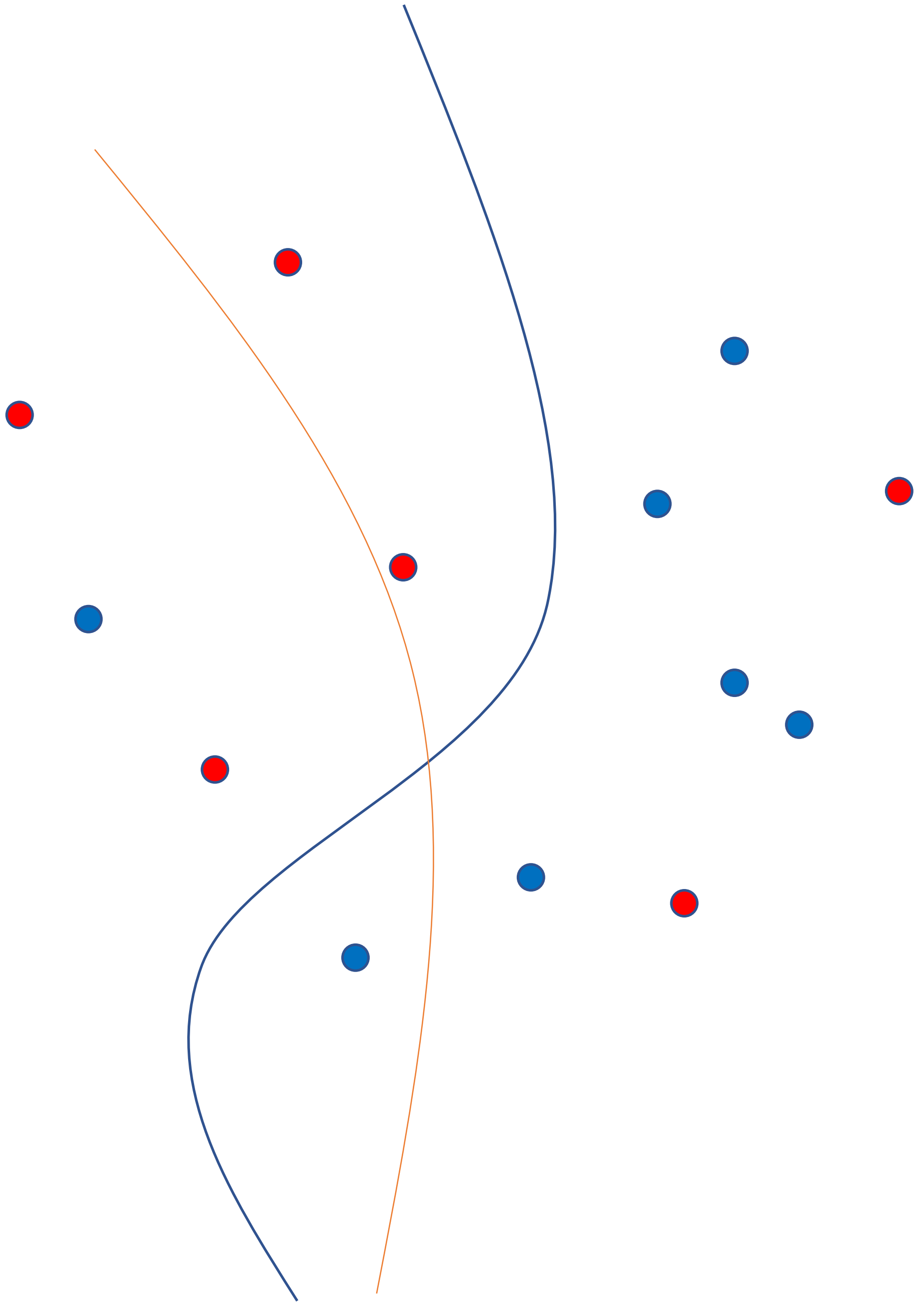

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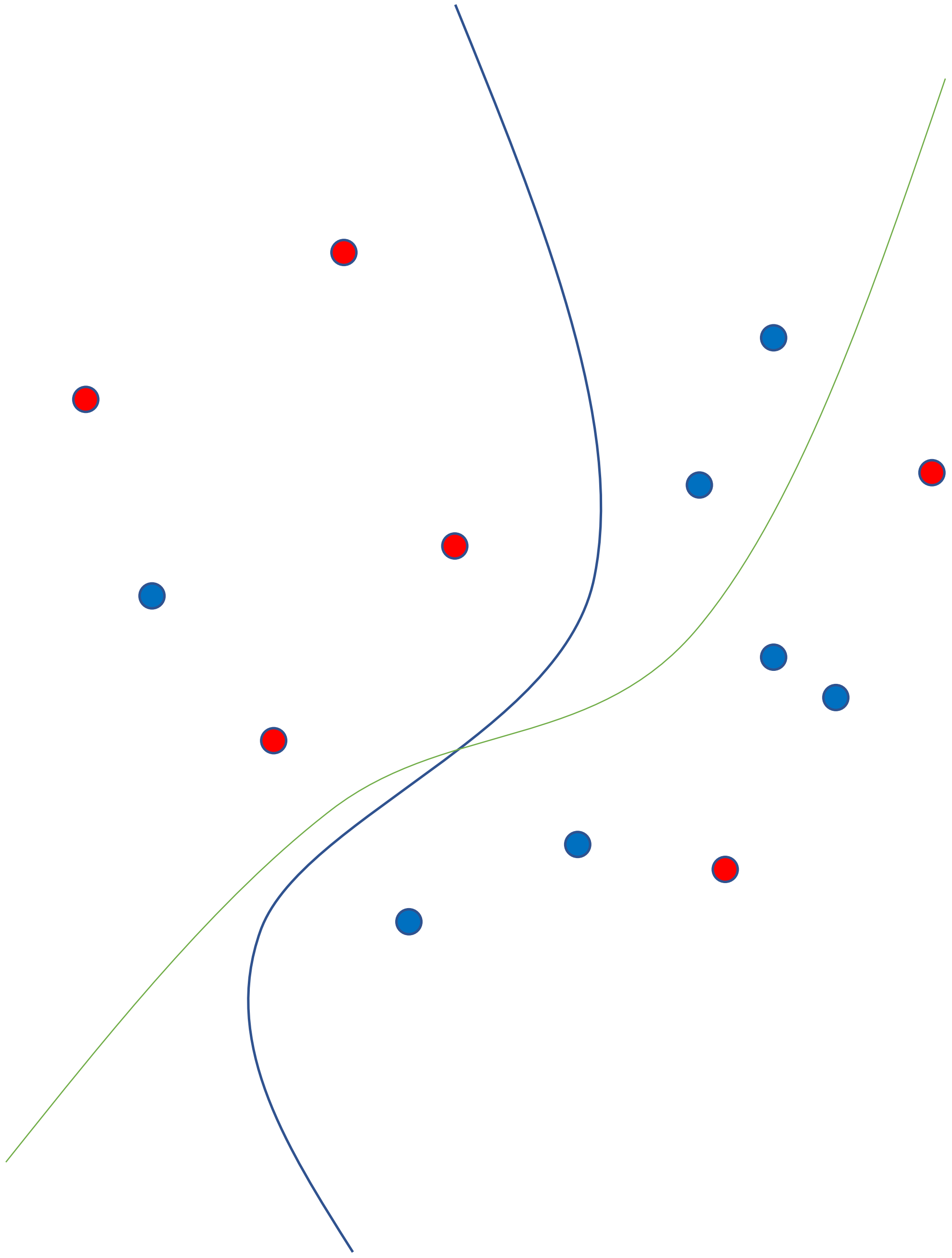

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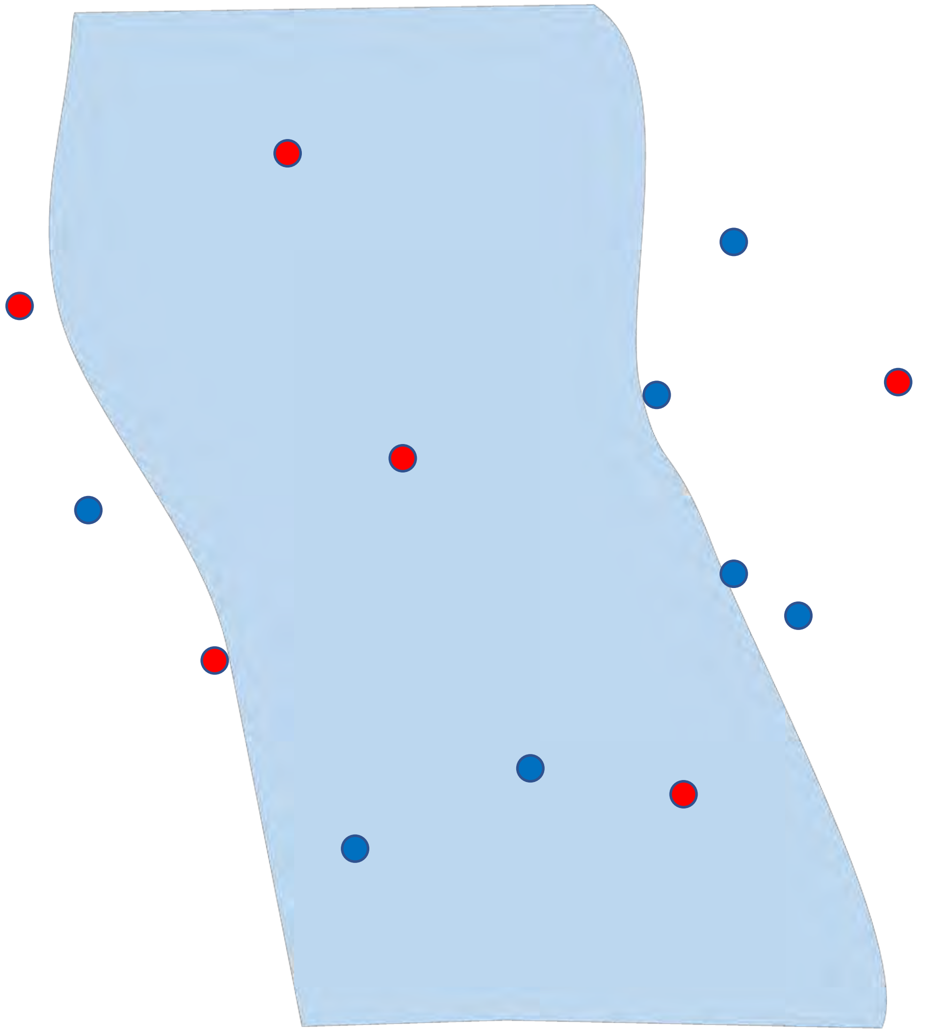

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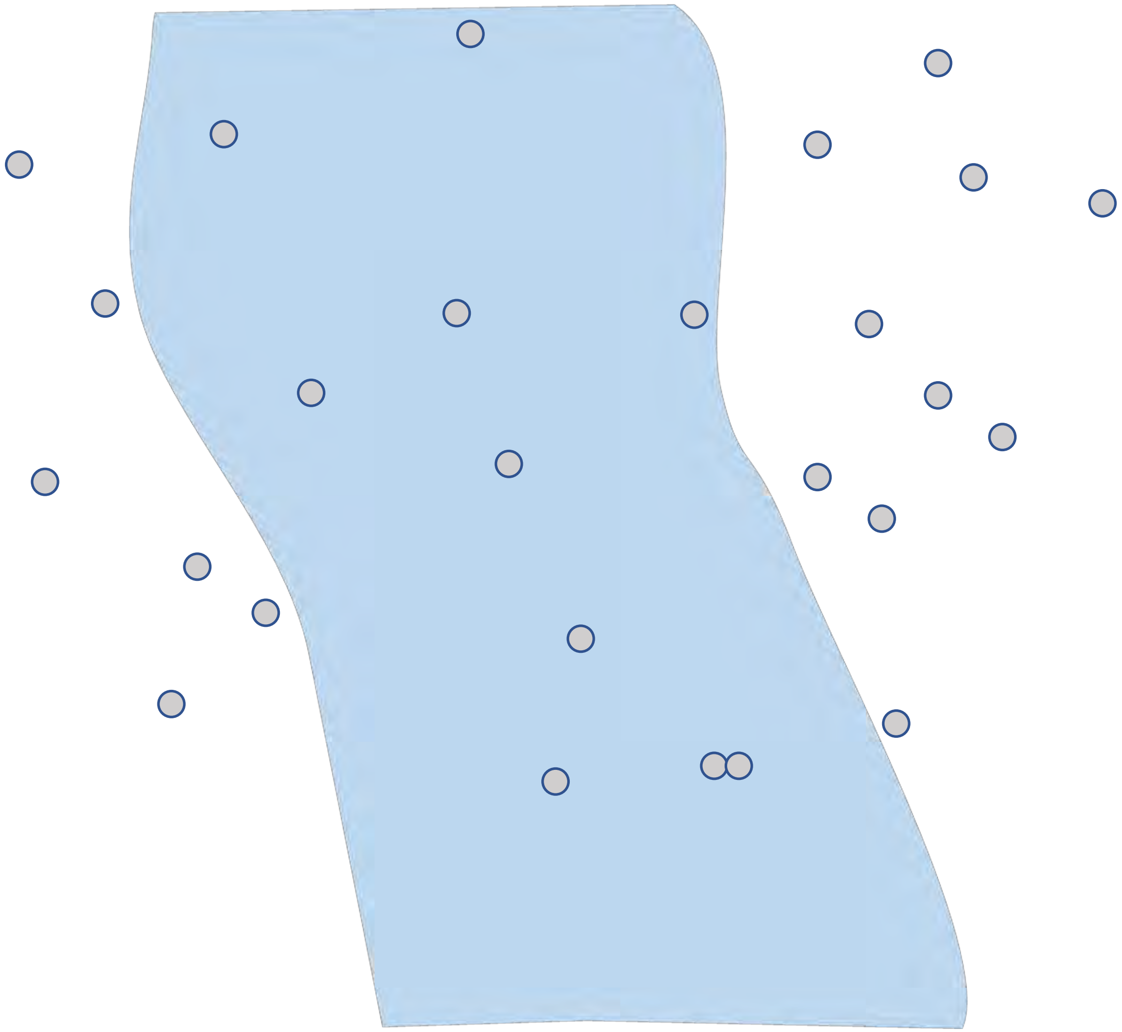

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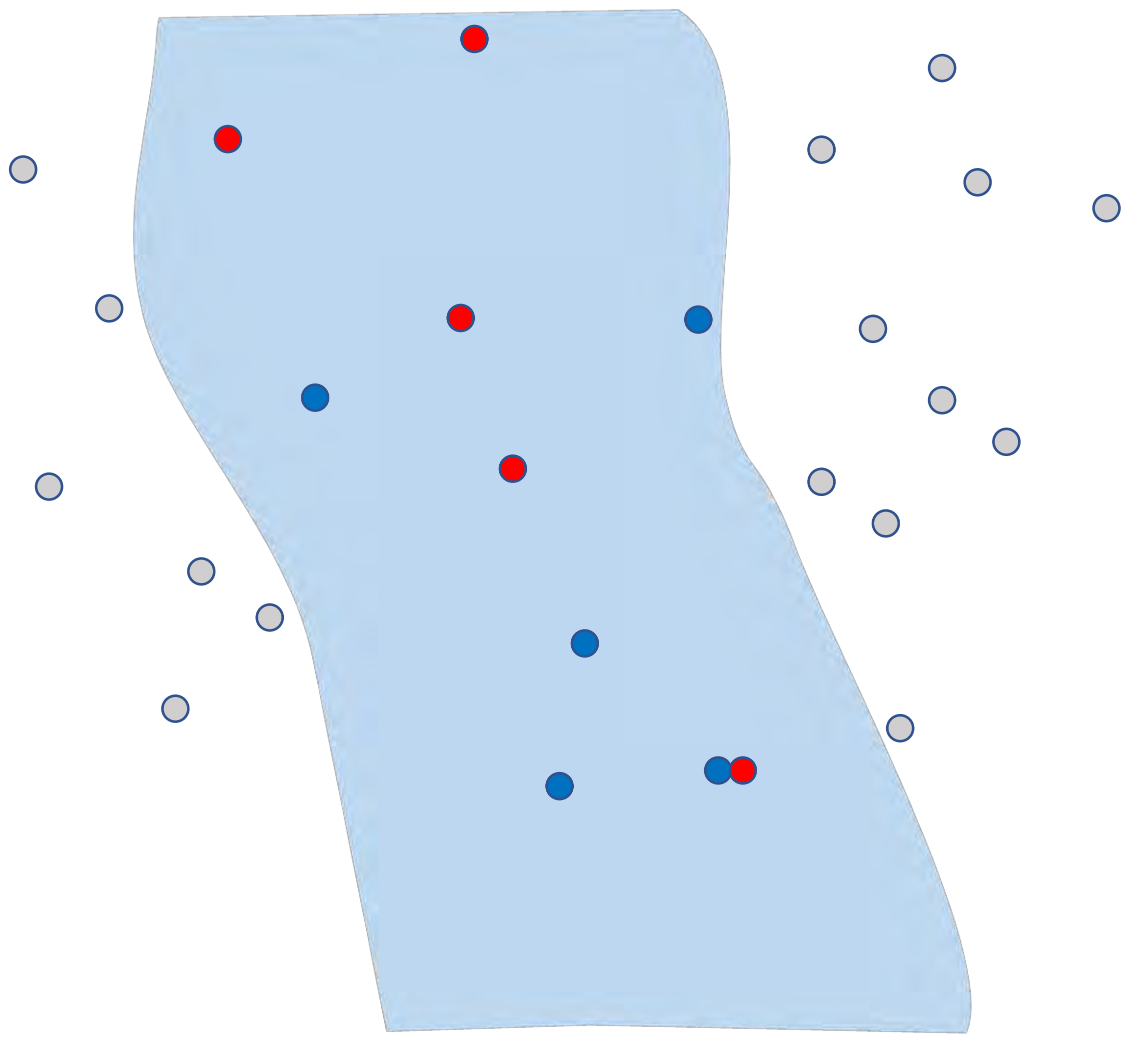

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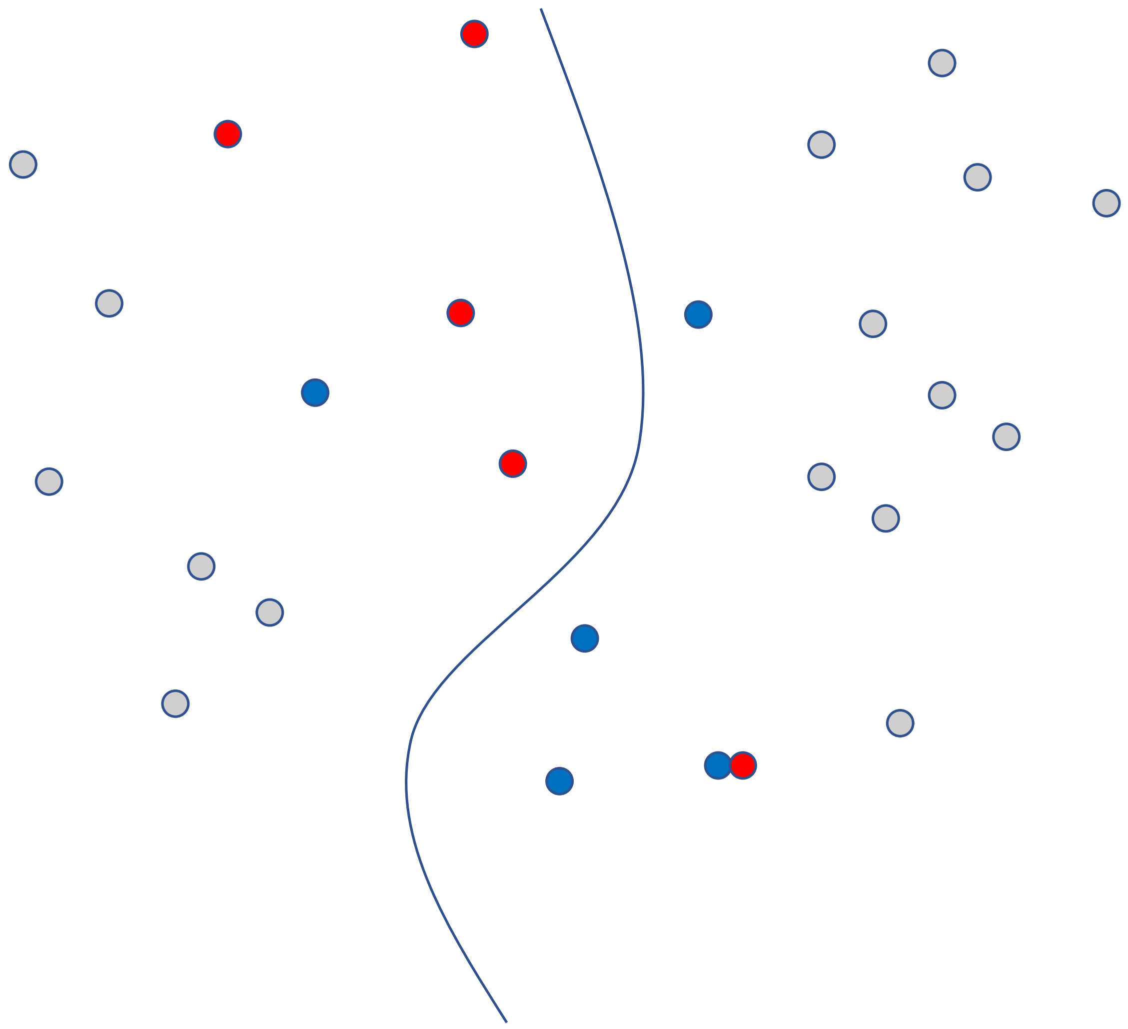

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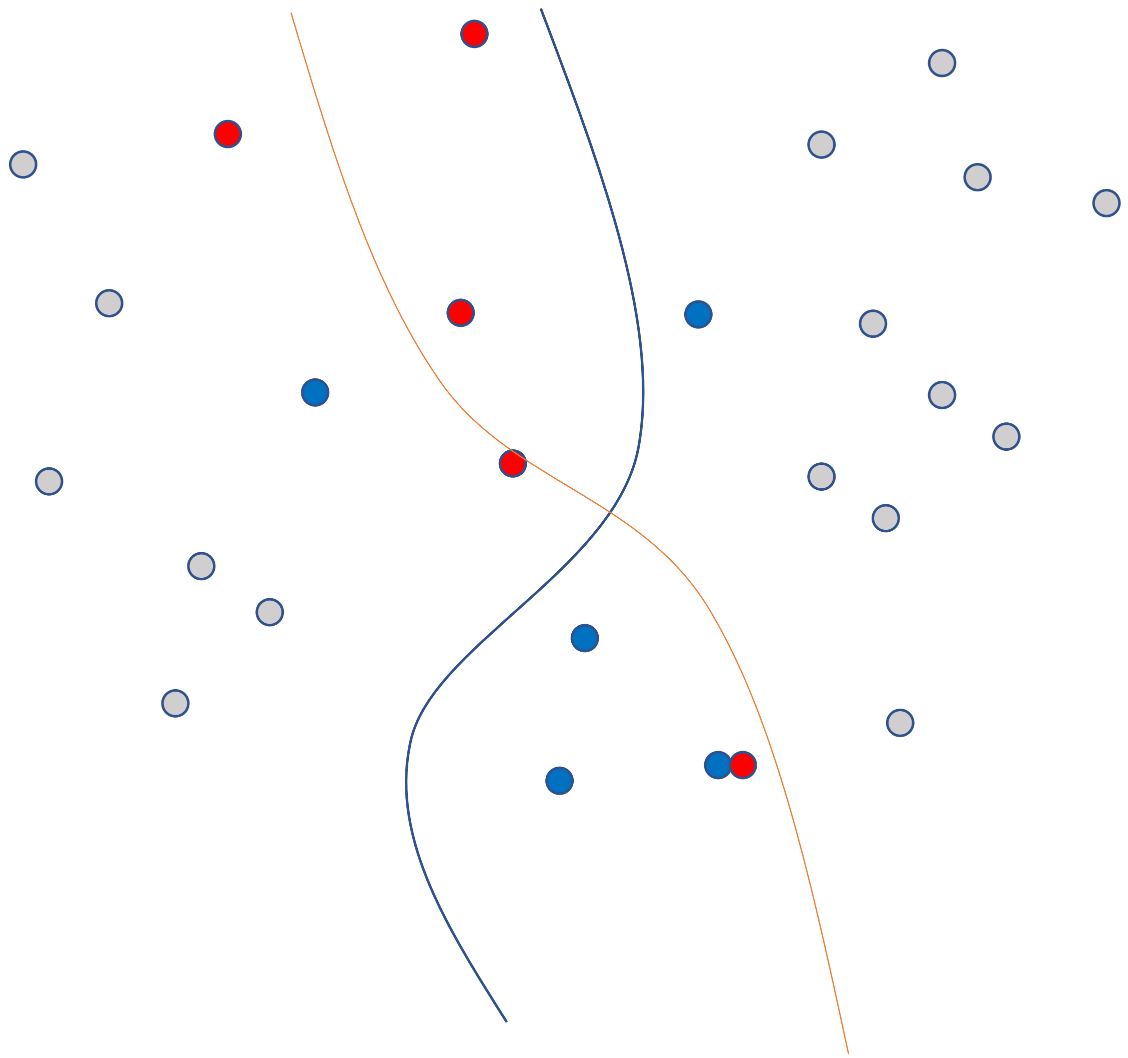
output final $\hat{f}$



Agnostic Active Learning

$$\text{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
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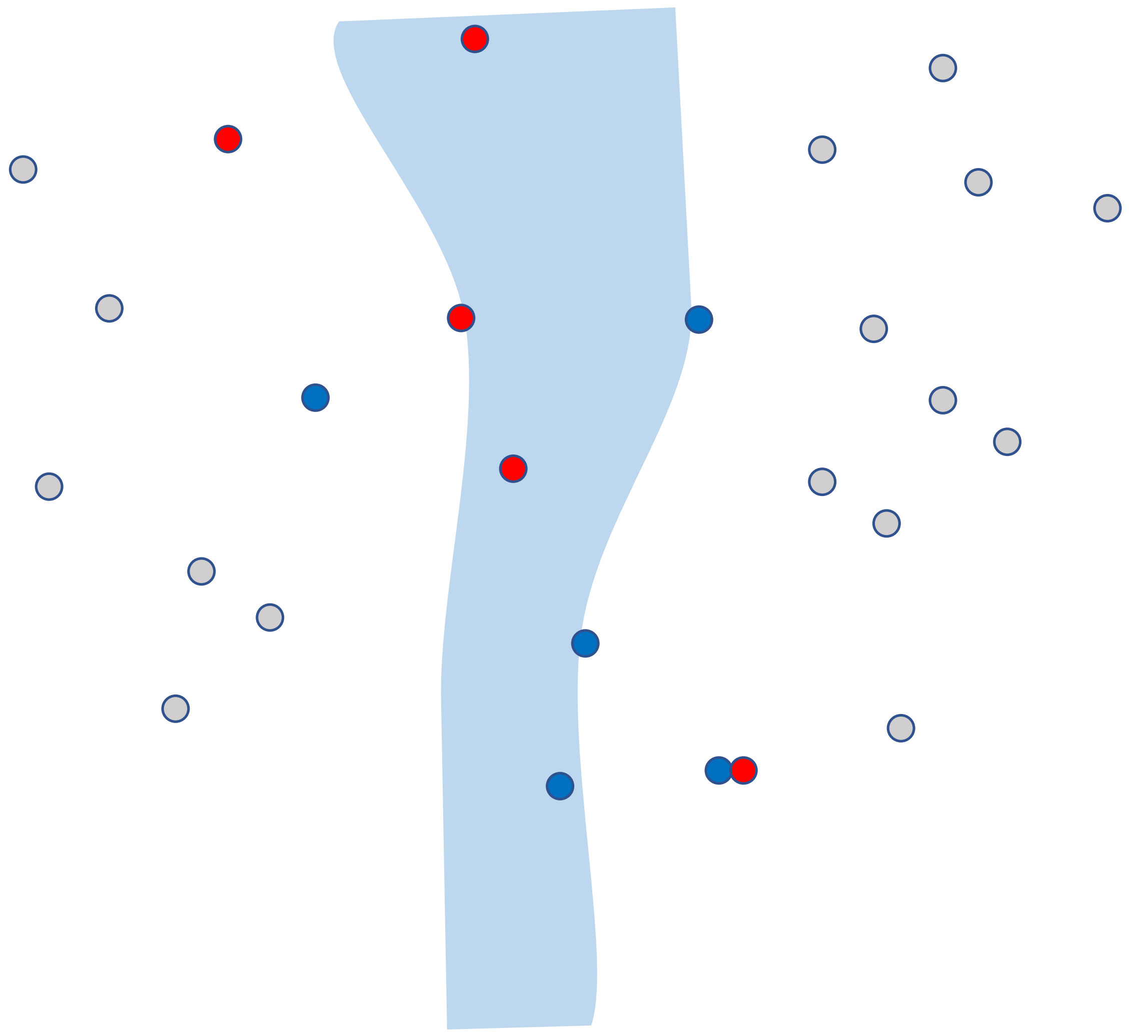

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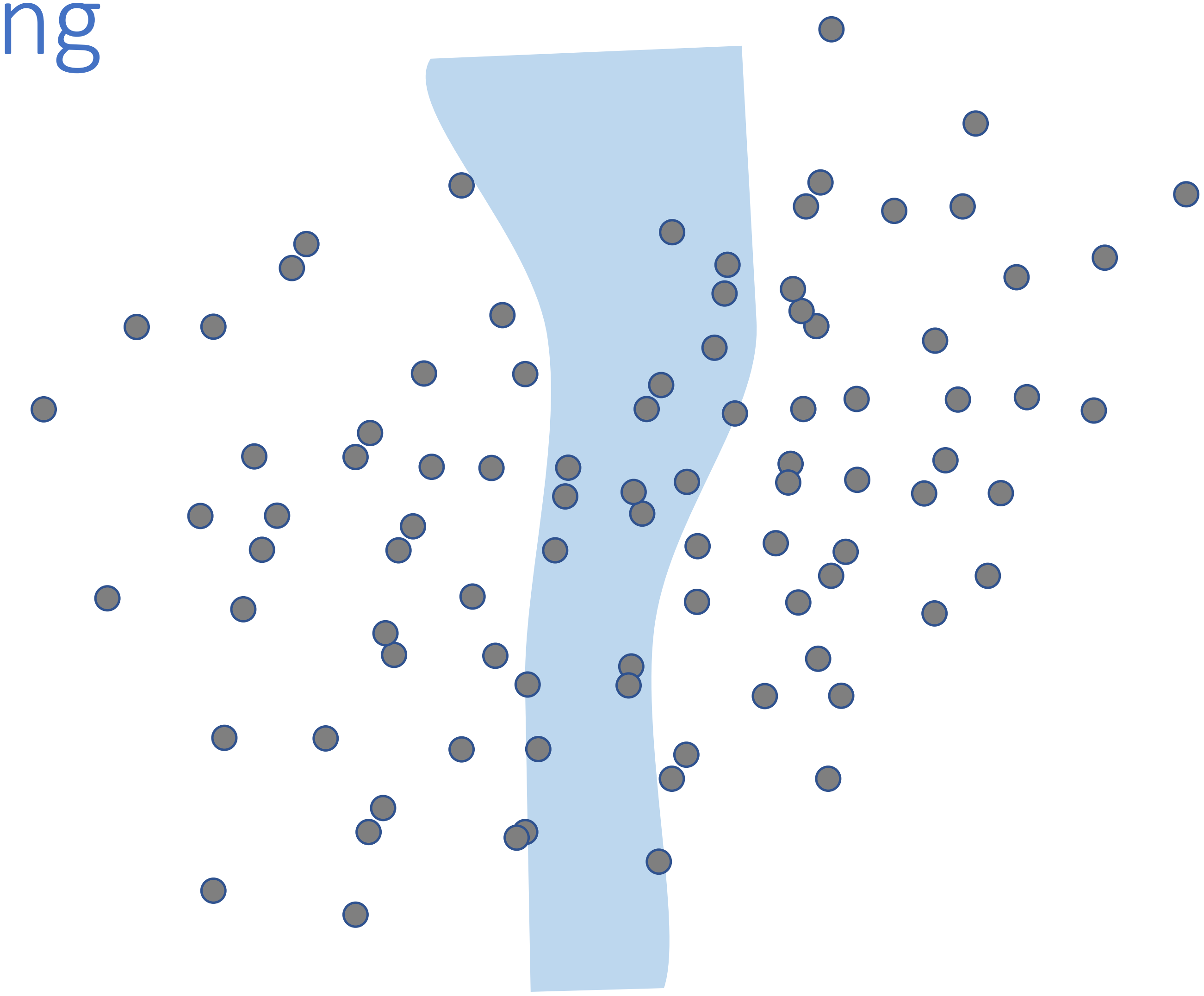

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output final $\hat{f}$

The point:

Any t with $f^* \in \mathcal{H}$ still,
$R(f^* | \mathrm{DIS}(\mathcal{H}))$ still **minimal** in $\mathcal{H}$

$\Rightarrow$

$$\hat{R}_Q(f^*) - \hat{R}_Q(\hat{f})$$
$$\leq R(f^* | \mathrm{DIS}(\mathcal{H})) - R(\hat{f} | \mathrm{DIS}(\mathcal{H})) + \sqrt{\hat{P}_Q(f^* \neq \hat{f}) \frac{d}{|Q|}}$$
$$\leq \sqrt{\hat{P}_Q(f^* \neq \hat{f}) \frac{d}{|Q|}}$$

$\Rightarrow$ **f^* never removed.**

Agnostic Active Learning

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$\Rightarrow$ **f^* never removed.**

Next: **How many labels does it use?**

Sample Complexity Analysis

Hanneke (2007,...)

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

$\mathrm{DIS}(\mathrm{B}(f^*, r)) := \{x \in \mathcal{X} : \exists f, f' \in \mathrm{B}(f^*, r), f(x) \neq f'(x)\}$

Disagreement coefficient:

$$\theta = \sup_{r > \epsilon} \frac{P_X(\mathrm{DIS}(\mathrm{B}(f^*, r)))}{r}$$

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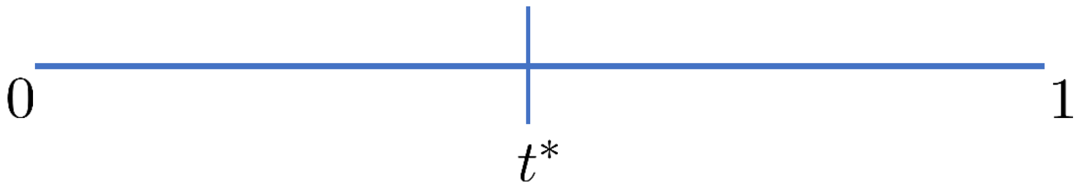
Example: **Thresholds**, P_X Uniform$(0, 1)$

$f(x) = \mathbb{I}[x \geq t]$



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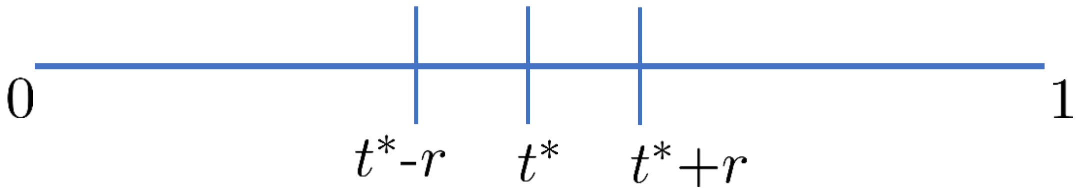

$\mathrm{DIS}(\mathrm{B}(f^*, r)) = [t^* - r, t^* + r)$

$P_X(\mathrm{DIS}(\mathrm{B}(f^*, r))) = 2r$

$\theta = 2$

Sample Complexity Analysis

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

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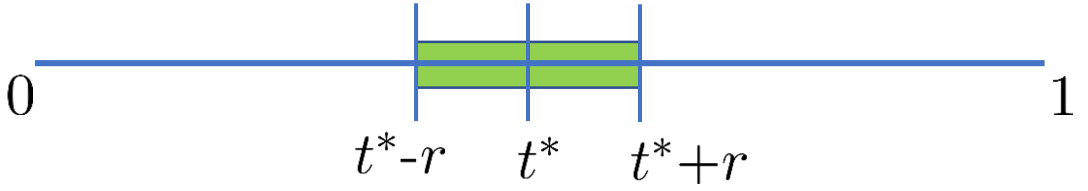

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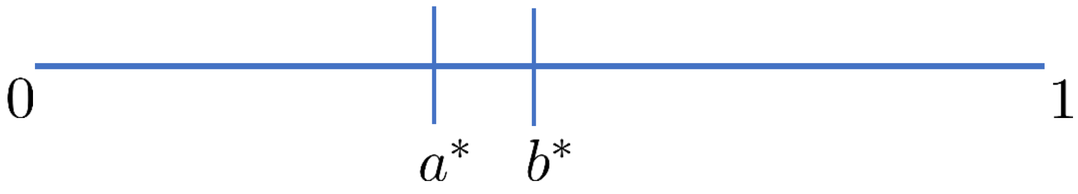

Sample Complexity Analysis

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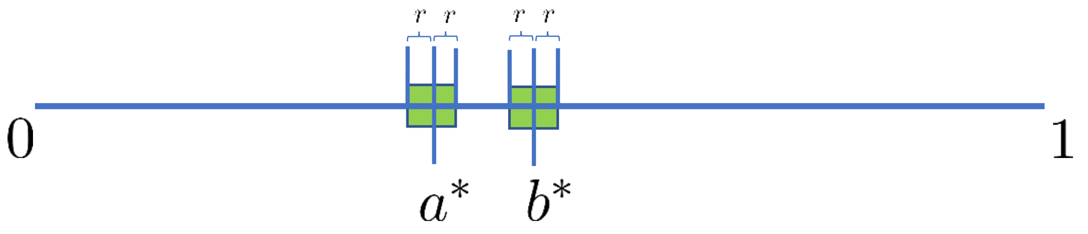

$w^* := b^* - a^*$

If $\boldsymbol{r < w^*}$,

$\mathrm{DIS}(\mathrm{B}(f^*, r)) = [a^* - r, a^* + r) \cup (b^* - r, b^* + r]$

$P_X(\mathrm{DIS}(\mathrm{B}(f^*, r))) = 4r$

Sample Complexity Analysis

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

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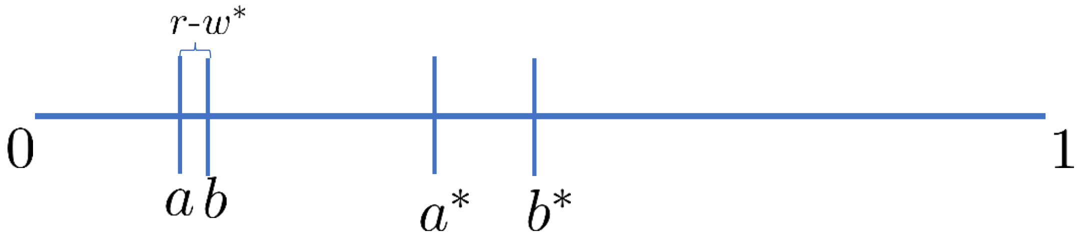

$w^* := b^* - a^*$

If $r > w^*$,

$\mathrm{DIS}(\mathrm{B}(f^*, r)) = \mathcal{X}$

$P_X(\mathrm{DIS}(\mathrm{B}(f^*, r))) = 1$

Sample Complexity Analysis

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

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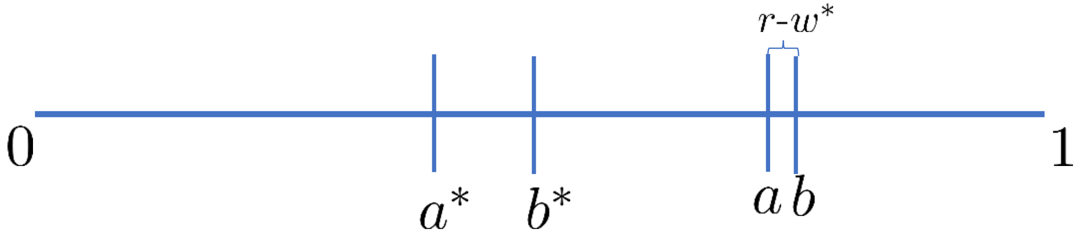

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Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

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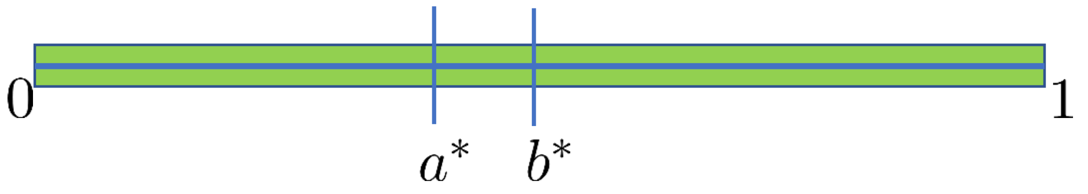

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If $\boldsymbol{r > w^*}$,

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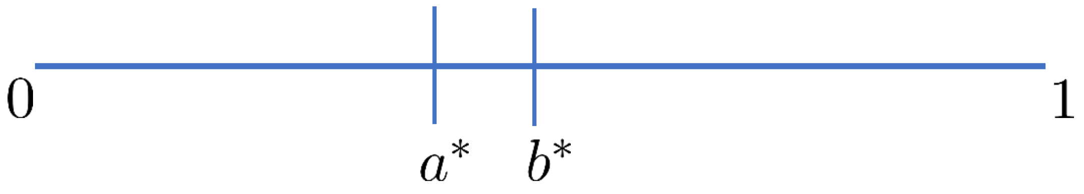

$w^* := b^* - a^*$

If $r < w^*$, $P_X(\mathrm{DIS}(\mathrm{B}(f^*, r))) = 4r$

If $r > w^*$, $P_X(\mathrm{DIS}(\mathrm{B}(f^*, r))) = 1$

$\Rightarrow \theta \leq \max\{4, \frac{1}{w^*}\}$

Sample Complexity Analysis

Ball: $\mathrm{B}(f^*, r) := \{f \in \mathcal{H} : P_X(f \neq f^*) \leq r\}$

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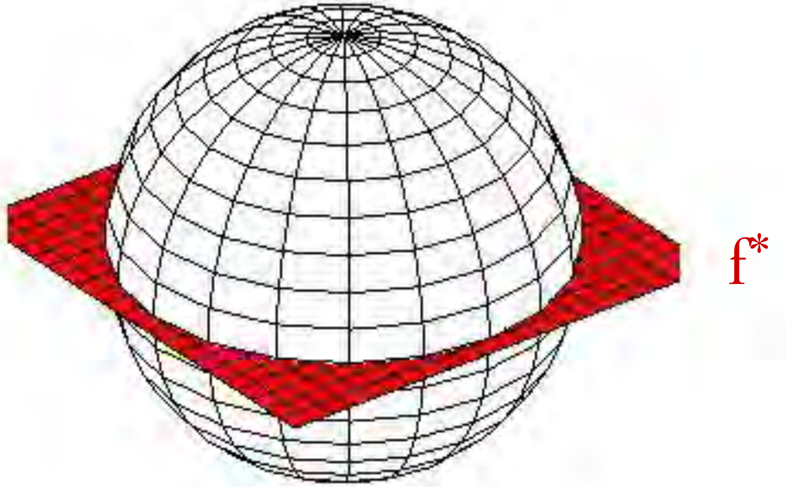
Example: homog. linear separators (bias 0), n dimensions, uniform P_X on sphere.



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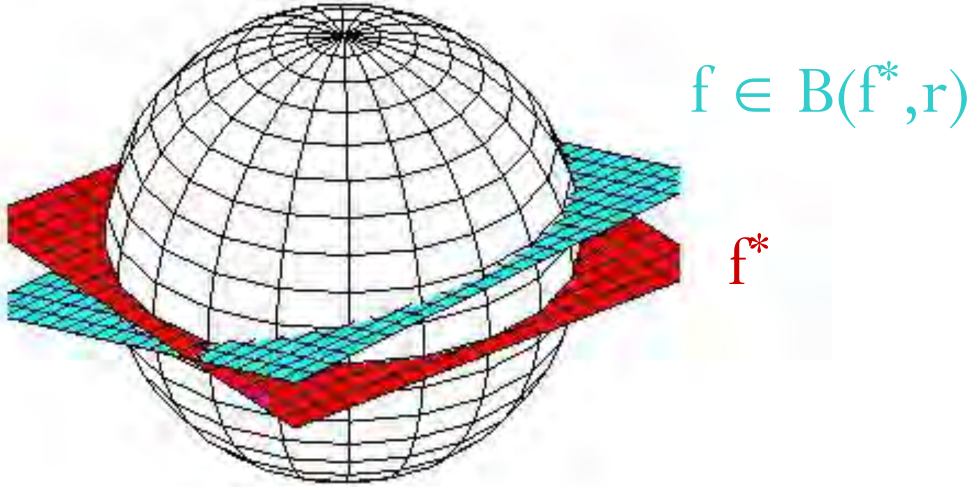

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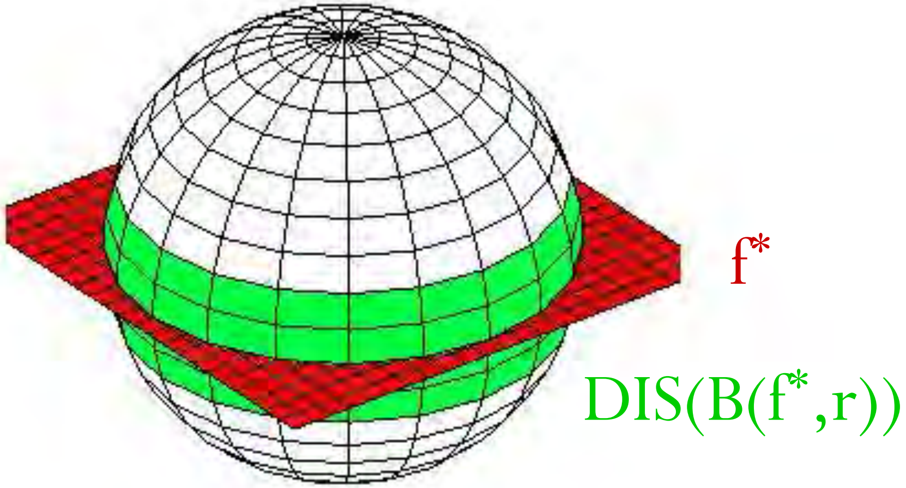

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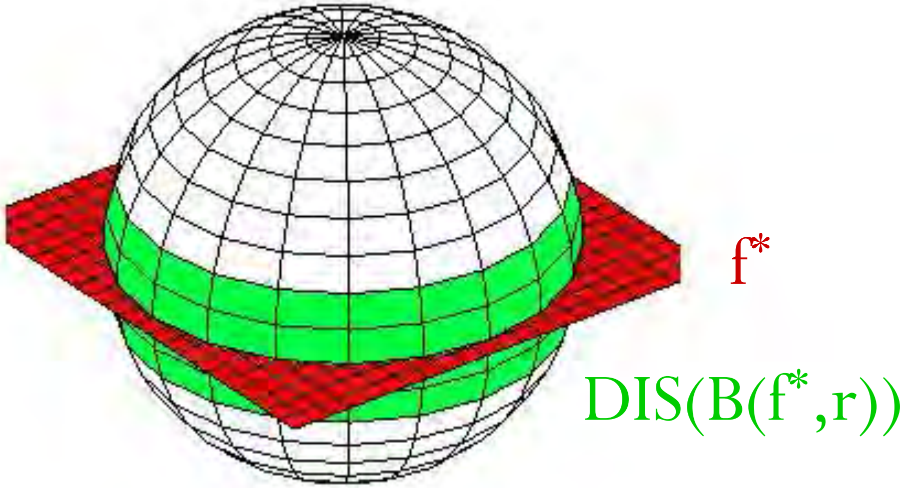

Some geometry $\Rightarrow$ for small r,

$P_X(DIS(B(f^*, r))) \propto \sqrt{n} r$.

$\Rightarrow \qquad \boldsymbol{\theta \propto \sqrt{n}}$.

Sample Complexity Analysis

Bounded Noise assumption: (aka Massart noise)

$$\exists \beta < 1/2 \text{ s.t. } P(Y \neq f^*(X)|X) \leq \beta \text{ everywhere}$$

	Sample Complexity: $R(\hat{f}) \leq R(f^*) + \epsilon$	Excess Error: n labels
Passive	$\frac{d}{\epsilon}$	$\frac{d}{n}$
Active	$d\theta \log(\frac{1}{\epsilon})$	$e^{-n/d\theta}$

Sample Complexity Analysis

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
2. **label** points in $Q = \mathrm{DIS}(\mathcal{H}) \cap S$
3. **optimize** $\hat{f} \leftarrow \operatorname{argmin}_{f \in \mathcal{H}} \hat{R}_Q(f)$
4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$.

output final $\hat{f}$

Theorem: $P(Y \neq f^*(X)|X) \leq \beta$. $R(\hat{f}) \leq R(f^*) + \epsilon$ with

$$\#\text{ labels} \approx d\theta \log(\tfrac{1}{\epsilon}).$$

Proof Sketch:

Round t, all $f \in \mathcal{H}$ **agree** on pts in $S \setminus Q$

Roughly, that means Step 4 only keeps f with
$R(f) - R(f^*) \lesssim \sqrt{P_X(f \neq f^*) \frac{d}{2^t}}$

$\Rightarrow$ surviving f after round t have $R(f) - R(f^*) \lesssim \frac{d}{2^t}$
$\Rightarrow t \gtrsim \log(\frac{d}{\epsilon})$ suffices

Also $\Rightarrow$ after round $t-1$, $\mathcal{H} \subseteq \mathrm{B}(f^*, d/2^{t-1})$

$$\Rightarrow |Q| \lesssim P_X(\mathrm{DIS}(\mathrm{B}(f^*, d/2^{t-1})))|S| \leq \theta \frac{d}{2^{t-1}} |S| = \theta d2$$

$$\sum_{t=1}^{\log(d/\epsilon)} \theta d = \theta d \log(\tfrac{d}{\epsilon})$$

□

Sample Complexity Analysis

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
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output final $\hat{f}$

Bounded noise:

$$R(f) - R(f^*) = \int_{f \neq f^*} (P(Y = f^*(X)|X) - P(Y \neq f^*(X)|X))\mathrm{d}P_X$$
$$\geq (1 - 2\beta)P_X(f \neq f^*)$$

Theorem: $P(Y \neq f^*(X)|X) \leq \beta$. $R(\hat{f}) \leq R(f^*) + \epsilon$ with

$$\#\text{ labels} \approx d\theta \log(\tfrac{1}{\epsilon}).$$

Proof Sketch:

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$$\sum_{t=1}^{\log(d/\epsilon)} \theta d = \theta d \log(\tfrac{d}{\epsilon})$$

□

Sample Complexity Analysis

Agnostic Learning: (no assumptions)

Denote $\beta = R(f^*)$

	Sample Complexity: $R(\hat{f}) \leq R(f^*) + \epsilon$	Excess Error: n labels
Passive	$d\frac{\beta}{\epsilon^2}$	$\sqrt{\frac{d\beta}{n}}$
Active	$d\theta\frac{\beta^2}{\epsilon^2}$	$\sqrt{\frac{d\beta^2\theta}{n}}$

Sample Complexity Analysis

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
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4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$.

output final $\hat{f}$

Theorem: $\beta = R(f^*)$. $R(\hat{f}) \leq R(f^*) + \epsilon$ with

$$\#\text{ labels} \approx d\theta \frac{\beta^2}{\epsilon^2}.$$

Proof Sketch:

Round t, all $f \in \mathcal{H}$ **agree** on pts in $S \setminus Q$

Roughly, that means Step 4 only keeps f with
$R(f) - R(f^*) \lesssim \sqrt{P_X(f \neq f^*) \frac{d}{2^t}}$

$\Rightarrow$ surviving f after round t have $R(f) - R(f^*) \lesssim \sqrt{\beta \frac{d}{2^t}} + \frac{d}{2^t}$

(Roughly) $\sqrt{\beta \frac{d}{2^t}}$

$\Rightarrow t \gtrsim \log(d \frac{\beta}{\epsilon^2})$ suffices

Also $\Rightarrow$ after round $t-1$, $\mathcal{H} \subseteq \mathrm{B}\left(f^*, 2\beta + \sqrt{\beta \frac{d}{2^{t-1}}}\right) \subseteq \mathrm{B}(f^*, 3\beta)$ (for large t)

$\Rightarrow |Q| \lesssim P_X(\mathrm{DIS}(\mathrm{B}(f^*, 3\beta)))|S| \lesssim \theta\beta|S| = \theta\beta 2^t$

$$\sum_{t=1}^{\log(d\beta/\epsilon^2)} \theta\beta 2^t \sim \theta d \frac{\beta^2}{\epsilon^2}$$

□

Sample Complexity Analysis

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
2. **label** points in $Q = \mathrm{DIS}(\mathcal{H}) \cap S$
3. **optimize** $\hat{f} \leftarrow \underset{f \in \mathcal{H}}{\mathrm{argmin}}\, \hat{R}_Q(f)$
4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$

output final $\hat{f}$

$$P_X(f \neq f^*) \leq R(f) + R(f^*) = 2\beta + R(f) - R(f^*)$$

Theorem: $\beta = R(f^*)$. $R(\hat{f}) \leq R(f^*) + \epsilon$ with

$$\#\ \text{labels} \approx d\theta \frac{\beta^2}{\epsilon^2}.$$

Proof Sketch:
Round t, all $f \in \mathcal{H}$ **agree** on pts in $S \setminus Q$

Roughly, that means Step 4 only keeps f with
$R(f) - R(f^*) \lesssim \sqrt{P_X(f \neq f^*)\frac{d}{2^t}}$

$\Rightarrow$ surviving f after round t have $R(f) - R(f^*) \lesssim \sqrt{\beta \frac{d}{2^t}} + \frac{d}{2^t}$

(Roughly) $\sqrt{\beta \frac{d}{2^t}}$

$\Rightarrow t \gtrsim \log(d\frac{\beta}{\epsilon^2})$ suffices

Also $\Rightarrow$ after round $t-1$, $\mathcal{H} \subseteq \mathrm{B}\left(f^*, 2\beta + \sqrt{\beta \frac{d}{2^{t-1}}}\right) \subseteq \mathrm{B}(f^*, 3\beta)$ (for large t)

$\Rightarrow |Q| \lesssim P_X(\mathrm{DIS}(\mathrm{B}(f^*, 3\beta)))|S| \lesssim \theta\beta|S| = \theta\beta 2^t$

$$\sum_{t=1}^{\log(d\beta/\epsilon^2)} \theta\beta 2^t \sim \theta d \frac{\beta^2}{\epsilon^2}$$

□

Sample Complexity Analysis

When is θ small?

- Linear separators, P_X has a density,
  f^* boundary intersects interior of support
  $\Rightarrow$ **θ bounded**

- Linear separators, P_X has a density
  $\Rightarrow \boldsymbol{\theta \ll \frac{1}{\epsilon}}$

- $\mathcal{H}$ smoothly-parametrized model,
  P_X "regular" density w/ compact support,
  other technical conditions on $\mathcal{H}$
  $\Rightarrow$ **$\theta \propto$ # parameters for $\mathcal{H}$**

- $\cdots$

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- …

Lots more →

Stopping Criterion

$$\mathrm{DIS}(\mathcal{H}) := \{x \in \mathcal{X} : \exists f, f' \in \mathcal{H}, f(x) \neq f'(x)\}$$

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1. **sample** 2^t unlabeled points S
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4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$.

output final $\hat{f}$

Stopping criteria:

- Any-time
- Label budget
- Run out of unlabeled data
- Check $\max_{f \in \mathcal{H}} \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}} < \epsilon$

Simpler Agnostic Active Learning

Hsu (2010,...)

$Q \leftarrow \{\}$

for $m = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** a random point x
2. **optimize** $\forall y,\ \hat{f}_y \leftarrow \underset{f \in \mathcal{H}: f(x)=y}{\operatorname{argmin}} \hat{R}_Q(f)$
3. if $|\hat{R}_Q(\hat{f}_+) - \hat{R}_Q(\hat{f}_-)| \leq \sqrt{\hat{P}_Q(\hat{f}_- \neq \hat{f}_+) \frac{d}{|Q|}}$

   then **label** x, add it to Q

output $\hat{f} = \underset{f \in \mathcal{H}}{\operatorname{argmin}} \hat{R}_Q(f)$

- Roughly same sample complexity as A^2.
- Can implement as a **reduction** to ERM.
- In practice, replace ERM with any passive learner.

Surrogate Loss

Hanneke & Yang (2012)

$Q \leftarrow \{\}$

for $m = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** a random point x
2. **optimize** $\forall y,\ \hat{f}_y \leftarrow \underset{f \in \mathcal{H}: f(x)=y}{\operatorname{argmin}} \hat{R}^{\ell}_Q(f)$
3. if $|\hat{R}_Q(\hat{f}_+) - \hat{R}_Q(\hat{f}_-)| \leq \sqrt{\hat{P}_Q(\hat{f}_- \neq \hat{f}_+) \frac{d}{|Q|}}$

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Consider learner that minimizes a **surrogate loss**
$\ell : \mathbb{R} \times \{-1, +1\} \to \mathbb{R}_+$
(e.g., hinge loss, squared loss, exponential loss, ...)

Now $\mathcal{H}$ is **real-valued** functions
$\hat{R}^{\ell}_Q(f) = \frac{1}{|Q|} \sum_{(x,y) \in Q} \ell(f(x), y)$

Theorem: Bounded noise, plus strong assumptions on $\mathcal{H}$,ℓ,P
still get $R(\hat{f}) \leq R(f^*) + \epsilon$ with # labels

$$\approx \theta d \log\left(\frac{1}{\epsilon}\right)$$

Importance-Weighted Active Learning

Beygelzimer, Dasgupta, Langford (2009)

$Q \leftarrow \{\}$

for $m = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** a random point x
2. **set** sampling probability p_x
3. **flip** coin with prob p_x of heads
4. if heads, **label** x, add to Q with weight $1/p_x$

output $\hat{f} = \operatorname{argmin}_{f \in \mathcal{H}} \hat{R}_Q(f)$ (weighted loss)

Use importance weights to stay **unbiased**:
$\mathbb{E}[\hat{R}_Q(f)] = R(f)$

Now Q set of triples (x, y, w)

$$\hat{R}_Q(f) = \frac{1}{|Q|} \sum_{(x,y,w) \in Q} w \mathbb{I}[f(x) \neq y]$$

- **Any** choice of Step 2 (setting p_x) is fine (just p_x not too small, else high variance)

- Can set p_x in a way to recover A^2 sample complexity
$p_x = \mathbb{I}\left[\, |\hat{R}_Q(\hat{f}_+) - \hat{R}_Q(\hat{f}_-)| \leq \sqrt{\hat{P}_Q(\hat{f}_+ \neq \hat{f}_-)\frac{d}{|Q|}}\,\right]$

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- In practice, replace ERM with any passive learner (e.g., ERM with a surrogate loss)
- (approx) implementation in **Vowpal Wabbit** library

Questions?

Further reading:

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Part 3: Beyond Disagreement-Based Active Learning – Current Directions

- Subregion-Based Active Learning
- Margin-Based Active Learning: Linear Separators
- Shattering-Based Active Learning
- Distribution-Free Analysis, Optimality
- TicToc: Adapting to Heterogeneous Noise
- Tsybakov Noise

Tutorial on Active Learning: Theory to Practice

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ICML | 2019
Thirty-sixth International Conference on Machine Learning

Subregion-Based Active Learning

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Subregion-based Active Learning

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
2. **label** points in $Q = \mathcal{R}_{\epsilon'_t}(\mathcal{H}) \cap S$
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Instead, pick **region $\mathcal{R}_{\epsilon'}(\mathcal{H})$** s.t.
$\forall f, f' \in \mathcal{H},\ P_X(x \notin \mathcal{R}_{\epsilon'}(\mathcal{H}) : f(x) \neq f'(x)) \leq \epsilon'$.

Pick ϵ' carefully each round,
$R(\hat{f}) - R(f^*) \leq \epsilon$ at end

e.g., Bounded noise: $\epsilon' \propto d2^{-t}$

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Pick region $\mathcal{R}_{\epsilon'}(\mathcal{H})$ s.t.
$\forall f, f' \in \mathcal{H},\ P_X(x \notin \mathcal{R}_{\epsilon'}(\mathcal{H}) : f(x) \neq f'(x)) \leq \epsilon'$.

$$\varphi_c := \sup_{r > \epsilon} \frac{P_X(\mathcal{R}_{r/c}(\mathrm{B}(f^*, r)))}{r}$$

Theorem: with **Bounded noise**,
$R(\hat{f}) \leq R(f^*) + \epsilon$ using # labels

$$\approx \varphi_c d \log\left(\frac{1}{\epsilon}\right)$$

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Agnostic case: $\varphi'_c := \sup_{r > \epsilon} \frac{P_X(\mathcal{R}_{r/c}(\mathrm{B}(f^*, 2\beta + r)))}{2\beta + r}$

Theorem:
$R(\hat{f}) \leq R(f^*) + \epsilon$ using # labels

$$\approx \varphi'_c d \frac{\beta^2}{\epsilon^2}$$

Subregion-Based Active Learning

Zhang & Chaudhuri, 2014

How to find such an $\mathcal{R}_{\epsilon'}(\mathcal{H})$?

- $\mathcal{R}_{\epsilon'}(\mathcal{H}) = \mathrm{DIS}(\mathcal{H})$ works
- Empirically (Zhang & Chaudhuri, 2014)
- Nice structure: e.g., **Linear separators**

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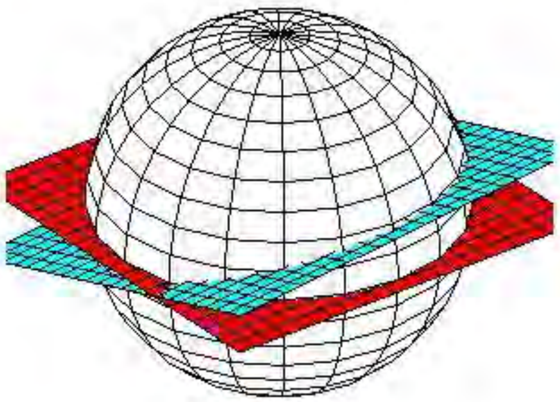
Uniform P_X on d-dim sphere

For $w \in \mathrm{B}(w^*, r)$, **project** to Span(w,w^*)



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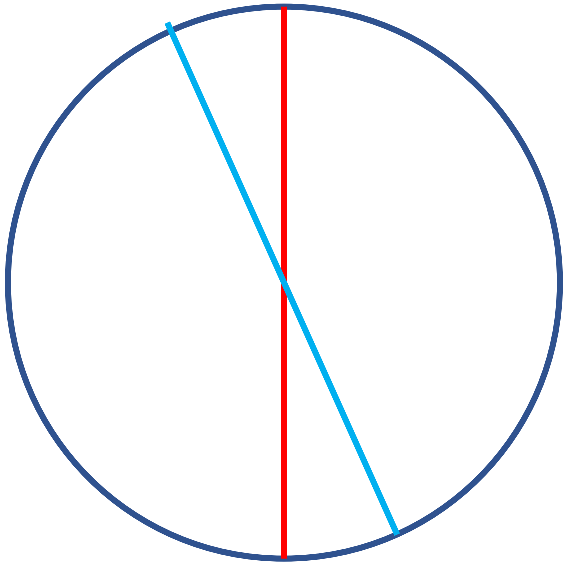
For $w \in \mathrm{B}(w^*, r)$, **project** to Span(w,w^*)

Most projected prob mass toward middle



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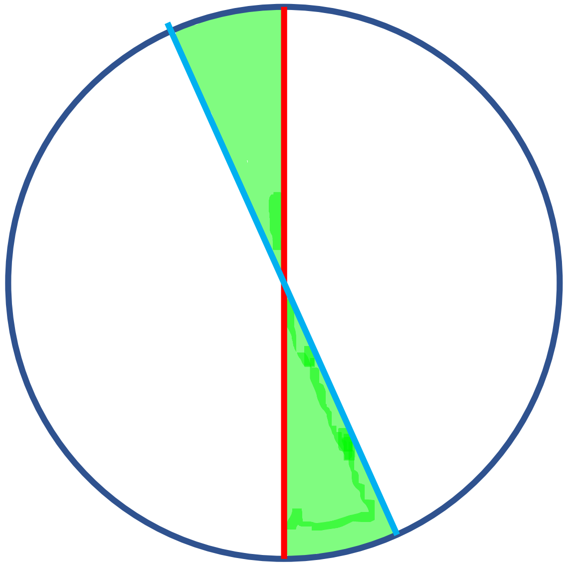

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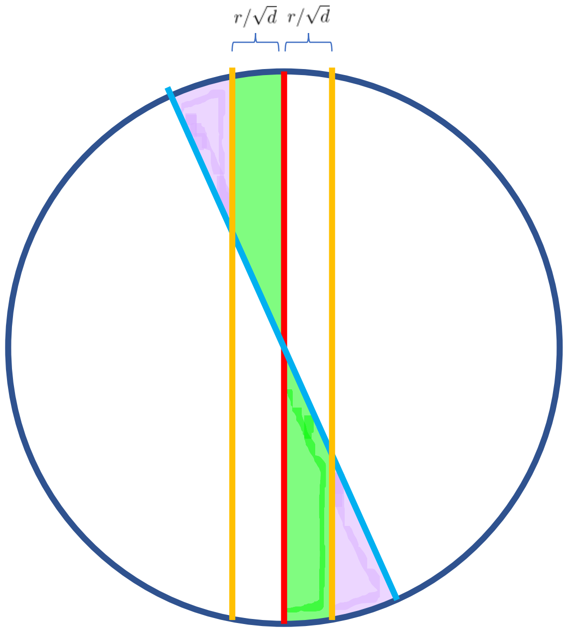

DIS($\{w, w^*\}$) in
slab of width $\approx r$

Most of its prob in
slab of width $\approx r/\sqrt{d}$

Subregion-Based Active Learning

How to find such an $\mathcal{R}_{\epsilon'}(\mathcal{H})$?

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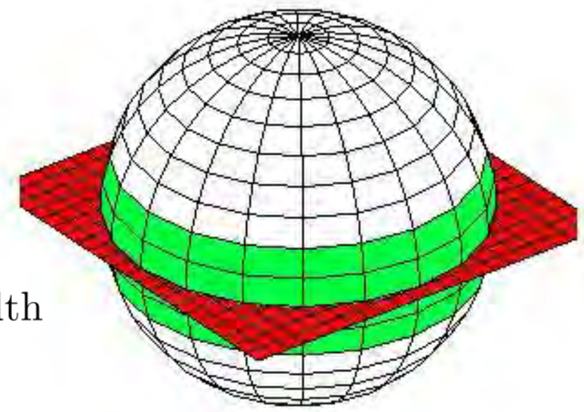
$\mathrm{DIS}(\mathrm{B}(f^*, r)) =$
slab of width $\approx r$

Take $\mathcal{R}_{r/c}(\mathrm{B}(f^*, r)) =$
slab of width $\approx r/\sqrt{d}$

Prob in slab $\approx \sqrt{d} \times$ width

$\Rightarrow \varphi_c \leq$ constant



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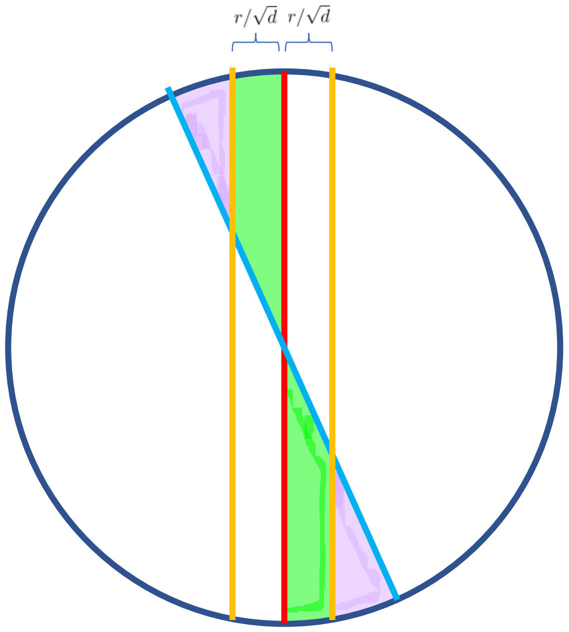

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DIS(B(f^*, r)) =
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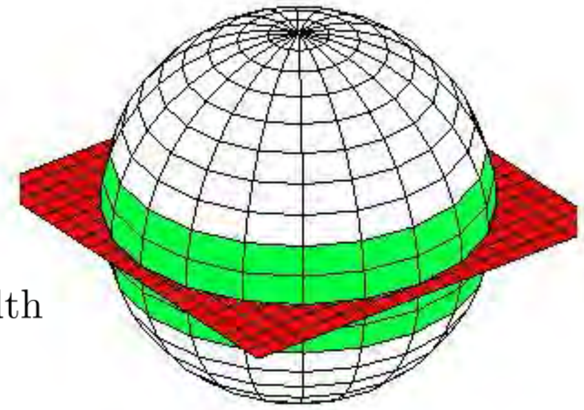

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Theorem: with **Bounded noise**,
$R(\hat{f}) \leq R(f^*) + \epsilon$ using # labels
$\approx \varphi_c d \log\left(\frac{1}{\epsilon}\right)$

$\Rightarrow$ # labels $\approx d \log(\frac{1}{\epsilon})$ suffice

Recall:
Passive $\approx \frac{d}{\epsilon}$

Comparison:
Recall $\theta \approx \sqrt{d}$
$\Rightarrow A^2$ # labels $\approx d^{3/2} \log(\frac{1}{\epsilon})$

Margin-Based Active Learning

(Balcan, Broder, Zhang, 2007; ...)

Margin-based Active Learning

Initialize $\hat{w}$

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** $d2^t$ unlabeled points S
2. **label** points in $Q =$ all $x \in S$ s.t. $<\hat{w}, x> \;\; \leq \;\; c2^{-t}/\sqrt{d}$
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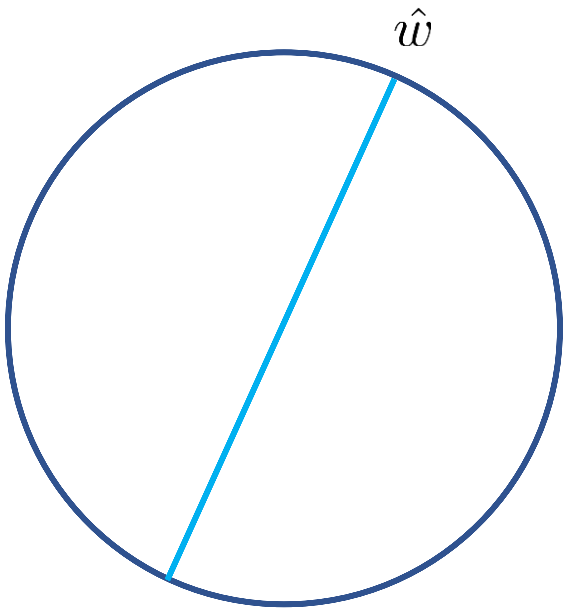

Uniform P_X on d-dim sphere

Margin-Based Active Learning

(Balcan, Broder, Zhang, 2007; ...)

Margin-based Active Learning

Initialize $\hat{w}$

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

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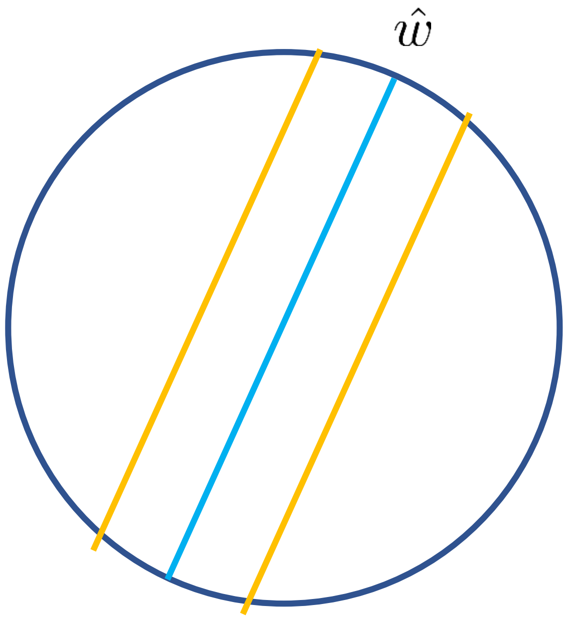

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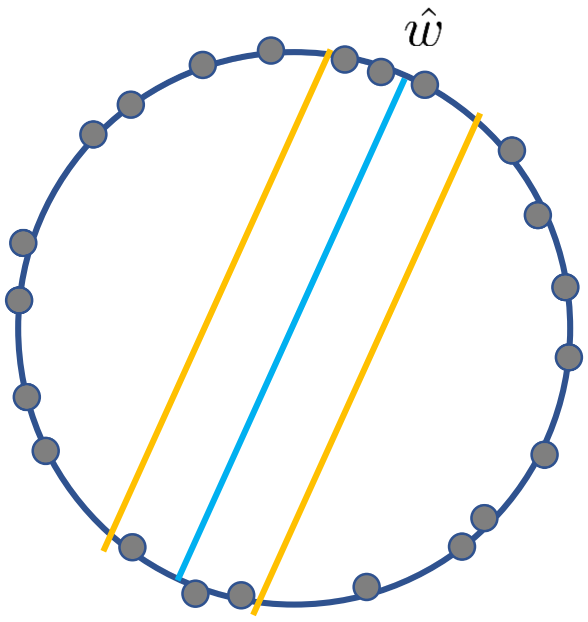

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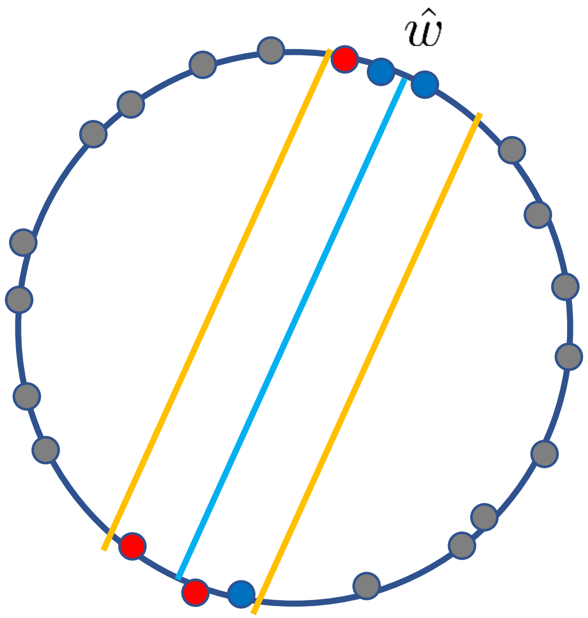

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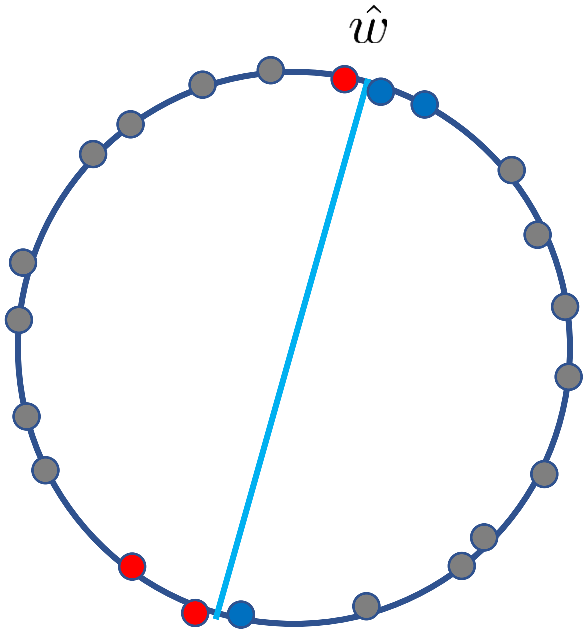

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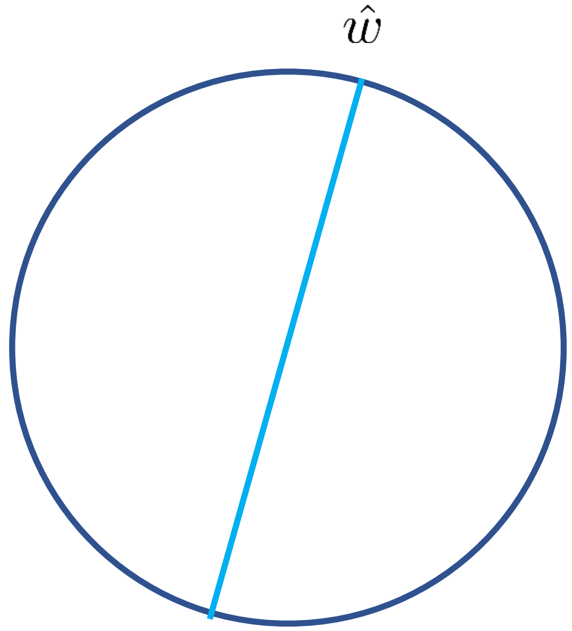


Uniform P_X on d-dim sphere

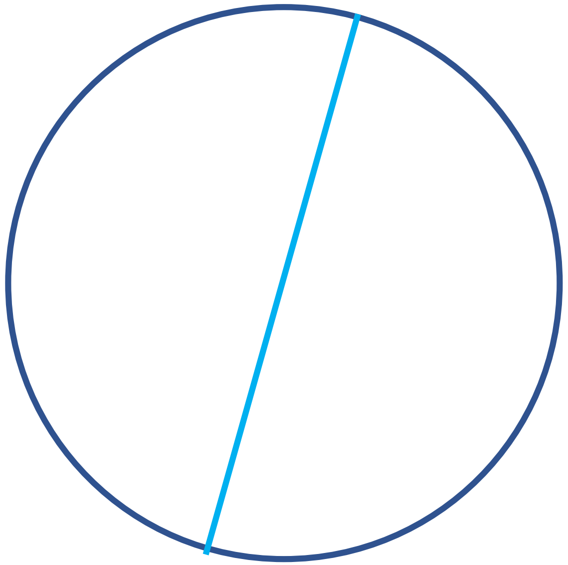
Theorem: with **Bounded noise**,
$R(\hat{f}) \leq R(f^*) + \epsilon$ using # labels
$\approx d \log\left(\frac{1}{\epsilon}\right)$

(also works for isotropic log-concave distributions)

Computational Efficiency

(Awasthi, Balcan, Long, 2014,...)

Uniform P_X on d-dim sphere



Efficient Alg

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Surrogate loss

$$\ell_t(<w, x>, y) \approx \max\{1 - 2^t\sqrt{d}(y<w, x>), 0\}$$

Hinge loss slope **changes** each round

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Theorem: with **Agnostic** case,
$R(\hat{f}) \le CR(f^*)$ **in polynomial time**

(was first alg. known to achieve these; even passively)

(also works for isotropic log-concave distributions)

Up Next:
Shattering-Based Active Learning

Shattering-Based Active Learning

(Hanneke, 2009, 2012)

Recall: $\mathcal{H}$ **shatters** $x_1, \ldots, x_k$ if all 2^k classifications realized by $\mathcal{H}$

$\mathrm{DIS}(\mathcal{H})$ checks for shattering 1 point.

Idea: Generalize to shattering ≥ 1 points.

Shattering-Based Active Learning

(Hanneke, 2009, 2012)

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A^2 (Agnostic Active)

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
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3. **optimize** $\hat{f} \leftarrow \underset{f \in \mathcal{H}}{\text{argmin}}\, \hat{R}_Q(f)$
4. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$.

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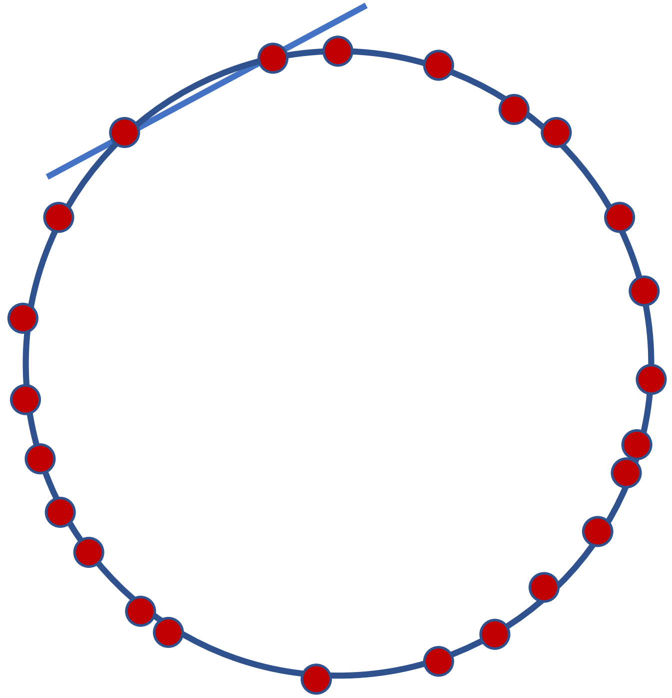
Example: Linear separators, Uniform P_X on circle

Suppose true labels are **all** -1

$\text{DIS}(\mathcal{H}) =$ **entire circle**



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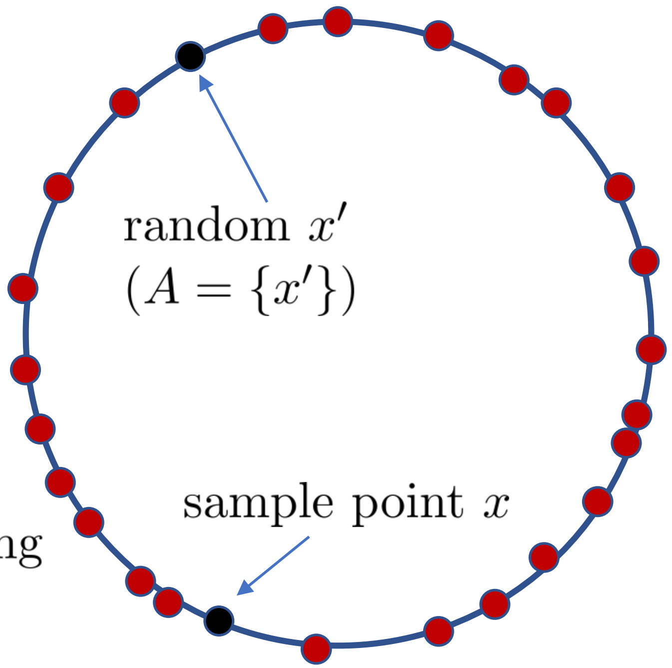
Try $k = 1$

Given sample x

Rand x' probably not close

Can't shatter $\{x, x'\}$ without a lot of points wrong

So won't query x



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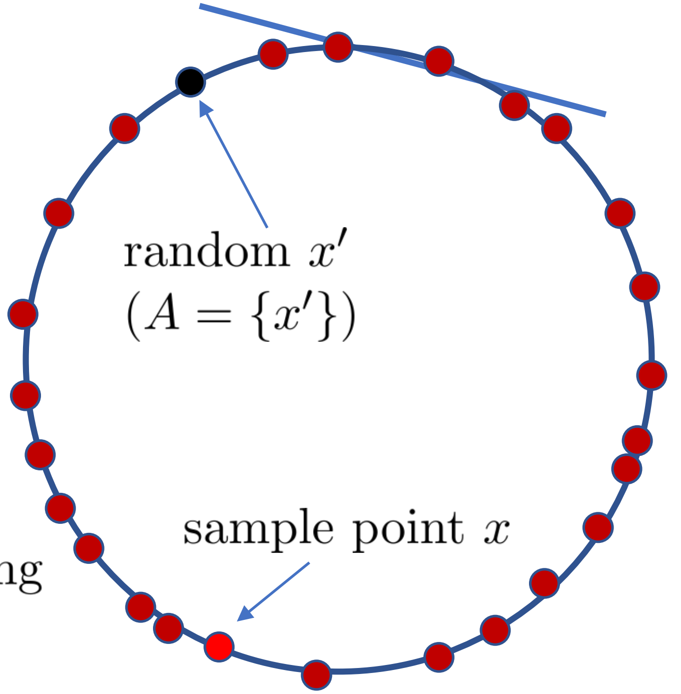

$\text{DIS}(\mathcal{H}_{x,-1})$ still entire circle (minus x)

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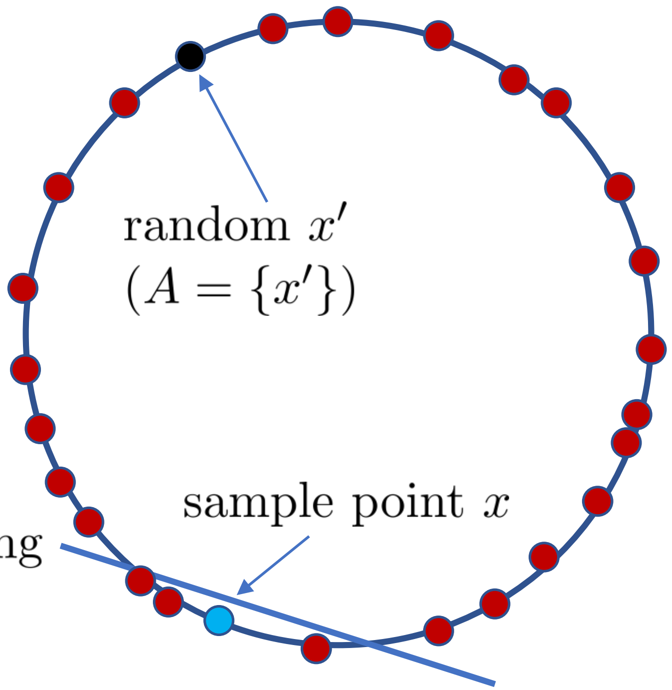

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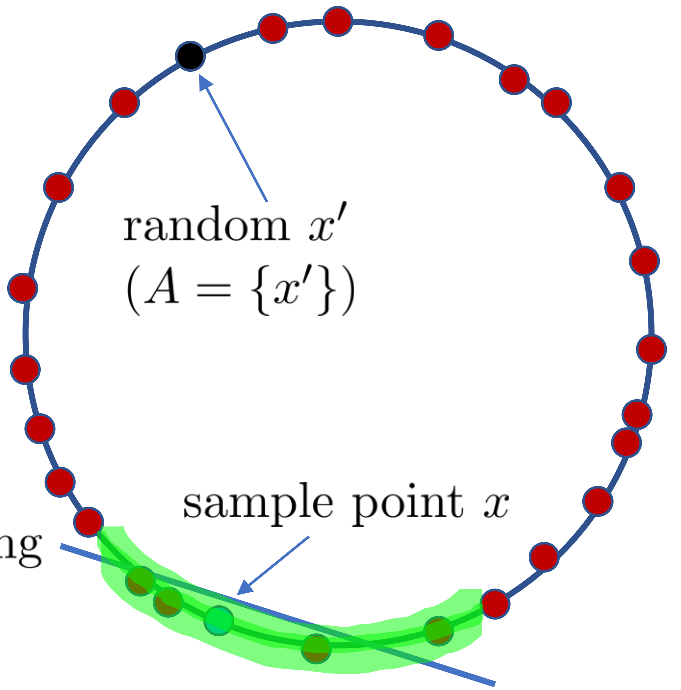

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Generally, need to try various k and pick one (See the papers)

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Generally, need to try various k and pick one (See the papers)

$$\theta^{(k)} := \sup_{r > \epsilon} \frac{P_X^k(A \in \mathcal{X}^k : \mathrm{B}(f^*, r) \text{ shatters } A)}{r}$$

$$\tilde{d} := \min\left\{k : P_X^k(A \in \mathcal{X}^k : \mathrm{B}(f^*, r) \text{ shatters } A) \xrightarrow[r \to 0]{} 0\right\}$$

$$\tilde{\theta} := \theta^{(\tilde{d})}$$

Theorem: For Bounded noise, $R(\hat{f}) \leq R(f^*) + \epsilon$ with # labels

$$\approx C \tilde{\theta} d \log\left(\frac{1}{\epsilon}\right)$$

Note: $\tilde{\theta} \ll \frac{1}{\epsilon}$

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Recall: $\mathcal{H}$ **shatters** $x_1, \ldots, x_k$ if all 2^k classifications realized by $\mathcal{H}$

Shattering-based Active Learning

for $t = 1, 2, \ldots$ (til *stopping-criterion*)

1. **sample** 2^t unlabeled points S
2. **label** points in $Q =$ all $x \in S$ s.t.
   $P_X^k(A \in \mathcal{X}^k : \mathcal{H} \text{ shatters } A \cup \{x\} | \mathcal{H} \text{ shatters } A) \geq \frac{1}{2}$
3. **add** the remaining points $x \in S$ to Q with label
   $\hat{y}_x := \operatorname{argmax}_y P_X^k(A \in \mathcal{X}^k : \mathcal{H}_{x,y} \text{ shatters } A | \mathcal{H} \text{ shatters } A)$
4. **optimize** $\hat{f} \leftarrow \operatorname{argmin}_{f \in \mathcal{H}} \hat{R}_Q(f)$
5. **reduce** $\mathcal{H}$: remove all f with $\hat{R}_Q(f) - \hat{R}_Q(\hat{f}) > \sqrt{\hat{P}_Q(f \neq \hat{f}) \frac{d}{|Q|}}$

output final $\hat{f}$

Denoting $\mathcal{H}_{x,y} := \{h \in \mathcal{H} : h(x) = y\}$

Generally, need to try various k and pick one (See the papers)

$$\theta^{(k)} := \sup_{r > \epsilon} \frac{P_X^k(A \in \mathcal{X}^k : \mathrm{B}(f^*, r) \text{ shatters } A)}{r}$$

$$\tilde{d} := \min\left\{k : P_X^k(A \in \mathcal{X}^k : \mathrm{B}(f^*, r) \text{ shatters } A) \xrightarrow[r \to 0]{} 0\right\}$$

$$\tilde{\theta} := \theta^{(\tilde{d})}$$

Theorem: For Bounded noise, $R(\hat{f}) \leq R(f^*) + \epsilon$ with # labels

$$\approx C \tilde{\theta} \tilde{d} \log\left(\tfrac{1}{\epsilon}\right)$$

Note: $\tilde{\theta} \ll \frac{1}{\epsilon}$

In the example: $\tilde{\theta} = 2$, $\theta = \frac{1}{\epsilon}$

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(may depend on f^*, P_X)

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Up Next:
Distribution-free Analysis

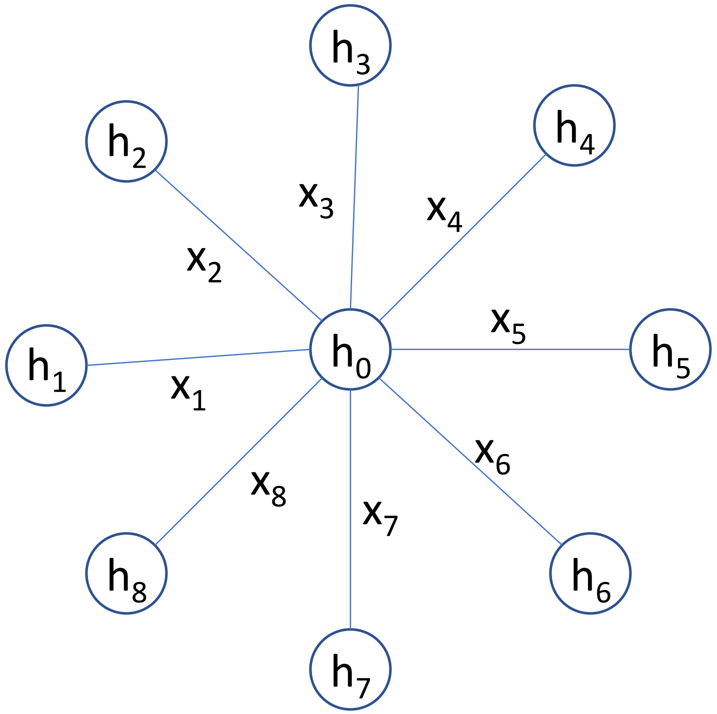
Distribution-Free Analysis

(Hanneke & Yang, 2015)

θ, φ, $\tilde{\theta}$ depend on f^*, P_X.

Can we do sample complexity analysis **without** distribution-dependence?

Definition: The **star number** $\mathfrak{s}$ is the largest k s.t. $\exists h_0, h_1, \ldots, h_k \in \mathcal{H}$, $\exists x_1, \ldots, x_k \in \mathcal{X}$ s.t. $\forall i \in \{1, \ldots, k\}$, $\{x_j : h_i(x_j) \neq h_0(x_j)\} = \{x_i\}$.



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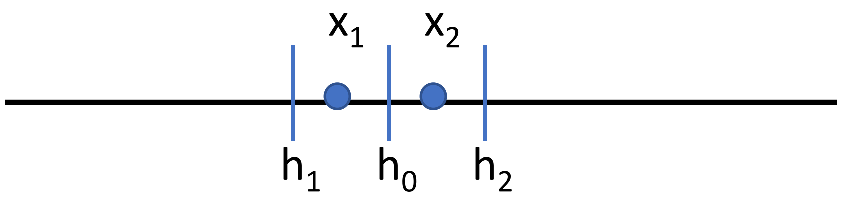
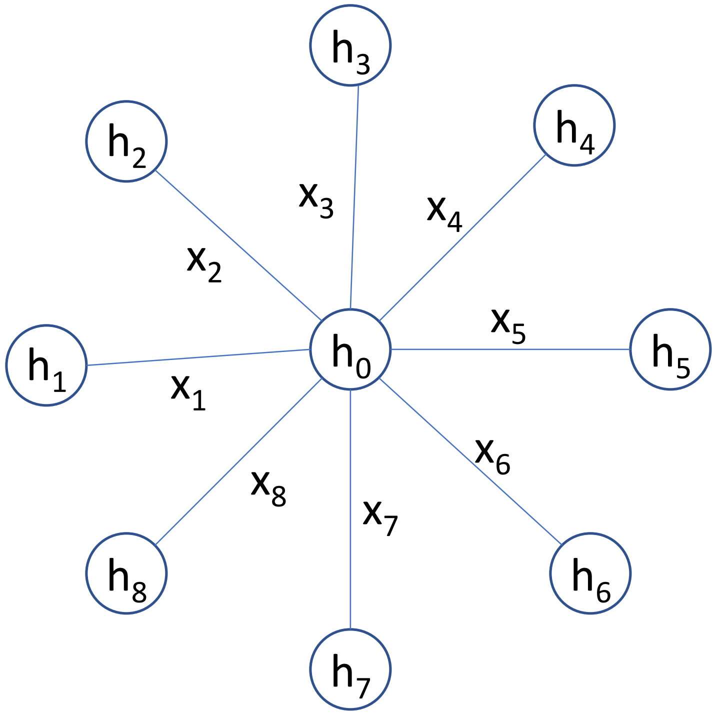
Example: Thresholds: $f(x) = \mathbb{I}[x \geq t]$.

$\mathfrak{s} = 2.$





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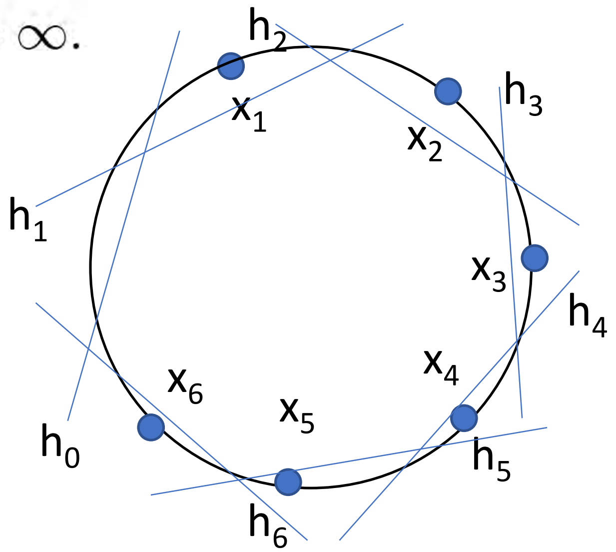
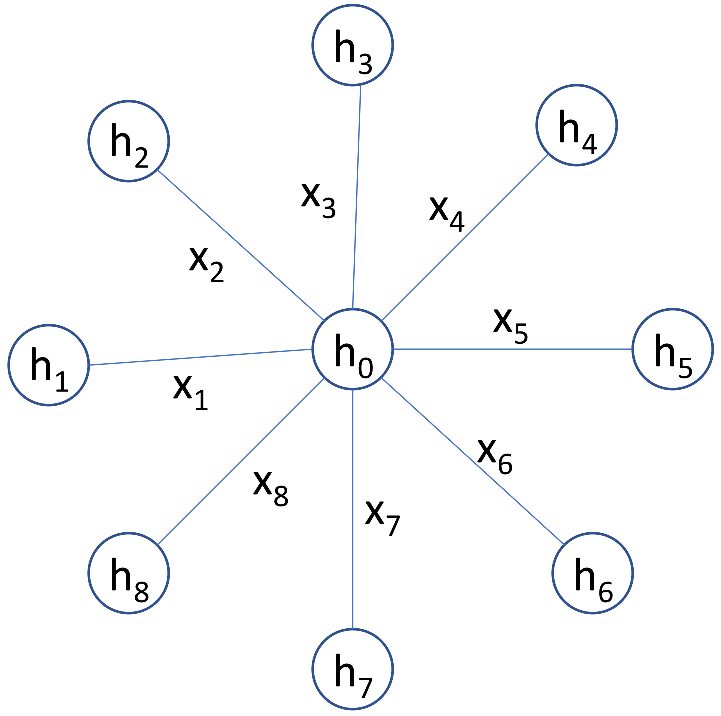
Example: Linear Separators in $\mathbb{R}^n$, $n \geq 2$:

$\mathfrak{s} = \infty$.





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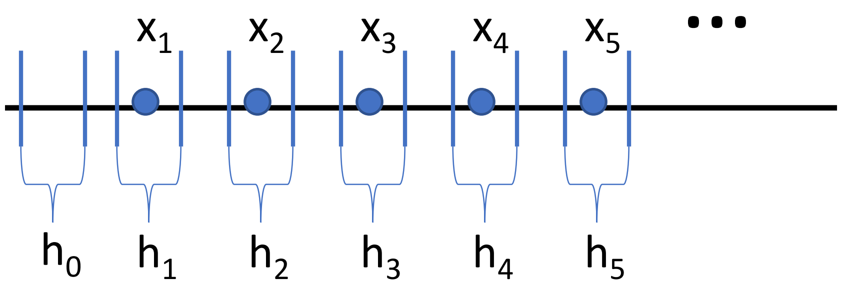
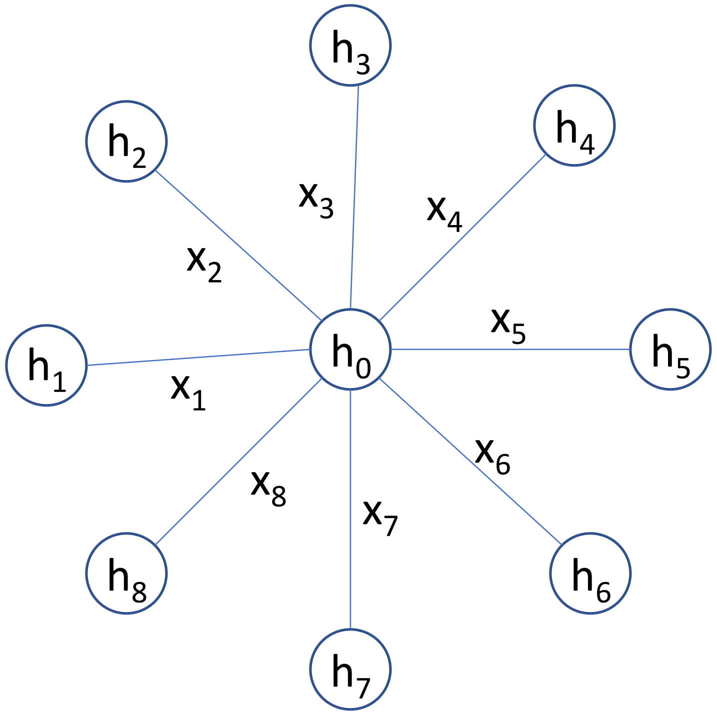
Example: Intervals: $x \mapsto \mathbb{I}[a \leq x \leq b]$

$\mathfrak{s} = \infty$.





Intervals of width w $(b - a = w > 0)$ on $\mathcal{X} = [0, 1]$: $\mathfrak{s} \approx \left\lfloor \frac{1}{w} \right\rfloor$.

Distribution-Free Analysis

(Hanneke & Yang, 2015; Hanneke, 2016)

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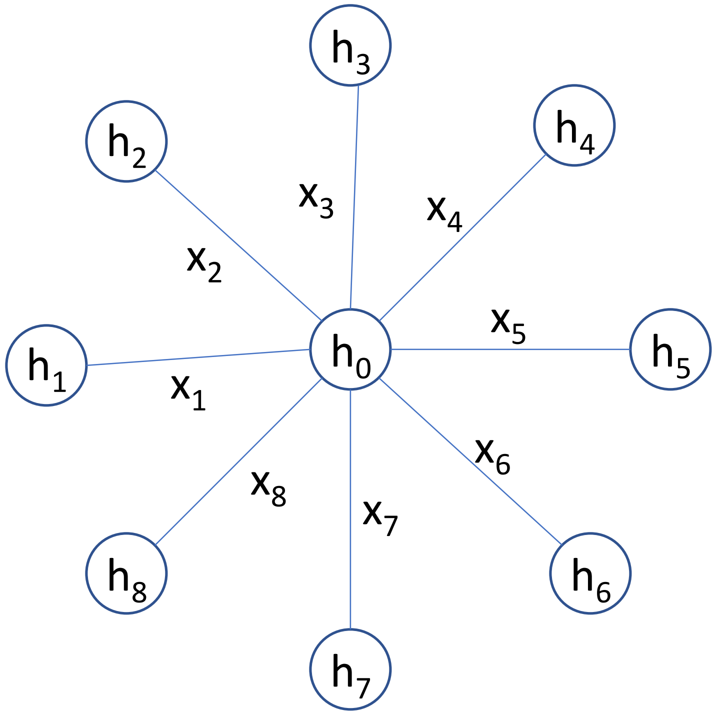

Theorem: $\sup_{P_X} \sup_{f^* \in \mathcal{H}} \theta = \sup_{P_X} \sup_{f^* \in \mathcal{H}} \varphi_c = \sup_{P_X} \sup_{f^* \in \mathcal{H}} \tilde{\theta} = \min\{\mathfrak{s}, \frac{1}{\epsilon}\} =: \mathfrak{s}_\epsilon$

Corollary:

Bounded noise # labels $\approx \mathfrak{s}_\epsilon d \log(\frac{1}{\epsilon})$

Agnostic ($\beta = R(f^*)$) # labels $\approx \mathfrak{s}_\beta d \frac{\beta^2}{\epsilon^2}$

Achieved by A^2

Distribution-Free Analysis

(Hanneke & Yang, 2015; Hanneke, 2016)

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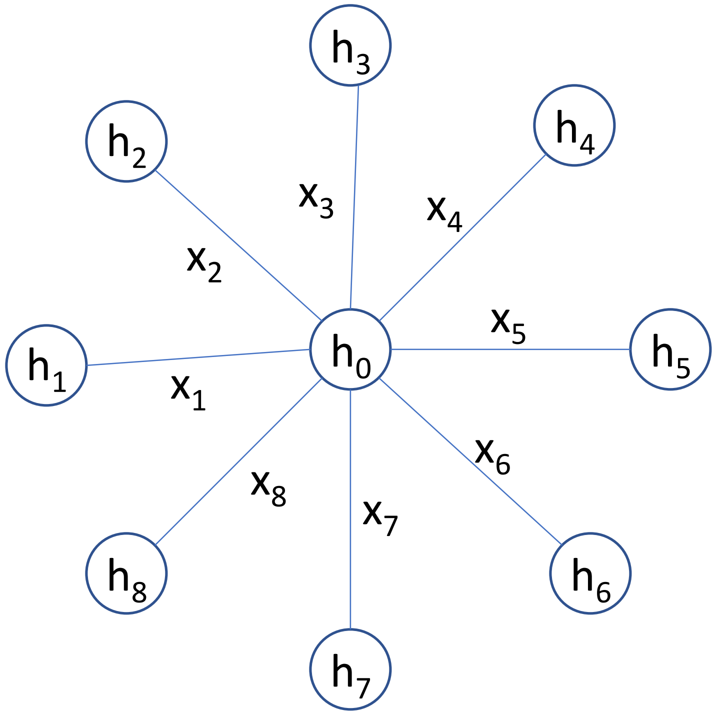
Different alg., Bounded noise
labels $\approx \mathfrak{s}_{\epsilon/d} \log(\frac{1}{\epsilon})$

Near-matching **lower bound**:
$\mathfrak{s}_\epsilon + d \log(\frac{1}{\epsilon})$



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Different alg., Bounded noise

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Near-matching **lower bound**:

$\mathfrak{s}_\epsilon + d \log(\frac{1}{\epsilon})$

Open Question:

Agnostic ($\beta = R(f^*)$)

labels

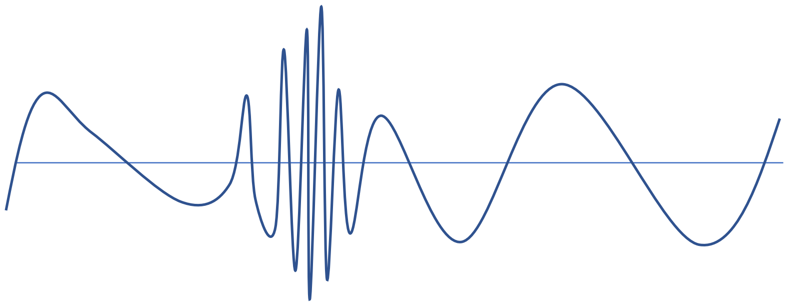
$\approx d\frac{\beta^2}{\epsilon^2} + \mathfrak{s}_{\epsilon/d} \log(\frac{1}{\epsilon})$ **?**

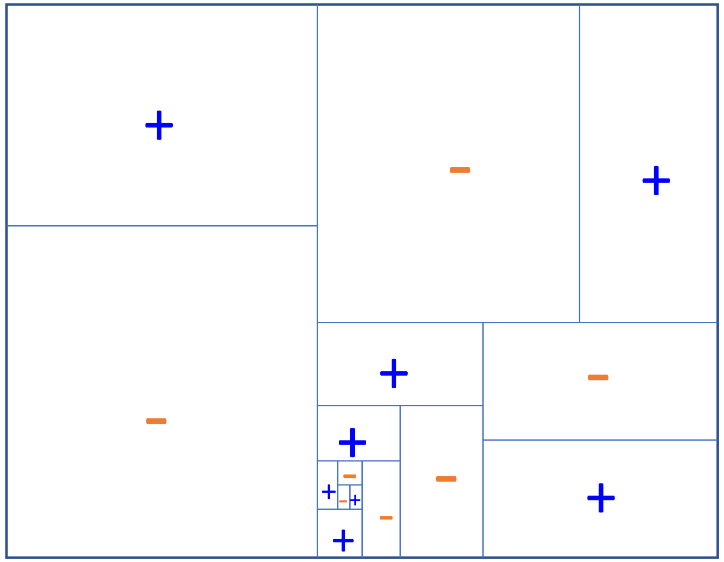
lower bound:

$d\frac{\beta^2}{\epsilon^2} + \mathfrak{s}_\epsilon + d \log(\frac{1}{\epsilon})$

Adapting to Heterogeneous Noise

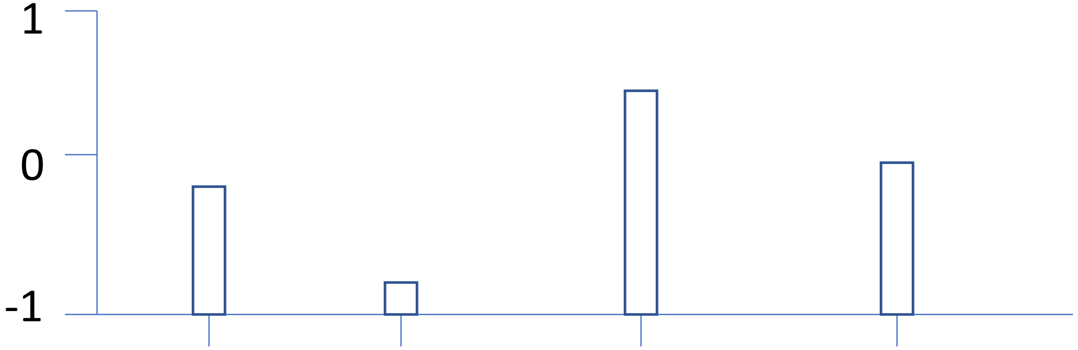
So far: Active learning for spatial heterogeneity of **opt function**:





Also consider: Spatial heterogeneity of **noise**:

$\eta(x) := \mathbb{E}[Y|X = x]$



Active Learning with TicToc

(Hanneke & Yang, 2015)

Algorithm: $\mathbb{A}(n)$
Input: Label budget n
Output: Classifier $\hat{f}_n$.

1. $\mathbb{L} \leftarrow \{\}$
2. For $m = 1, 2, \ldots$
3. $\quad X_{s_m} \leftarrow \textsc{GetSeed}(\mathbb{L}, m)$
4. $\quad \mathcal{L}_m \leftarrow \textsc{TicToc}(X_{s_m}, m)$
5. $\quad$ if $\mathcal{L}_m$ exists, $\mathbb{L} \leftarrow \mathbb{L} \cup \{(s_m, \mathcal{L}_m)\}$
6. $\quad$ If we've made n queries
7. $\quad\quad$ Return $\hat{f}_n \leftarrow \textsc{Learn}(\mathbb{L})$

An active learning alg. (e.g. A^2)

Main new part

A passive learning alg.

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7. Return $\hat{f}_n \leftarrow \text{LEARN}(\mathbb{L})$

Denote $\eta(x) = \mathbb{E}[Y|X = x]$
Suppose f^* is the **global** optimal function: $f^*(x) = \text{sign}(\eta(x))$

$\text{TICTOC}(\boldsymbol{X}, \boldsymbol{m})$:
Query X (or nearby) to try to guess $f^*(X)$
If can figure it out, return that label
If can't figure it out by τ_m queries give up (don't return a label)

Focus queries on less-noisy points.

Double advantage:

- Focusing on the points we actually care about:

$$R(f|x) - R(f^\star|x) = |\eta(x)|\mathbb{I}[f(x) \neq f^\star(x)]$$

(small $|\eta(x)| \Rightarrow$ not much effect on $R(f|x)$ if $f(x) = f^*(x)$ or not).

- And those points require fewer queries to determine $f^\star(X_i)$!

$\sim \frac{1}{\eta(X_i)^2}$ queries to determine $f^\star(X_i)$.

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Theorem: Bounded noise: # labels $\approx \mathfrak{s}_{\epsilon/d} \log(\frac{1}{\epsilon})$

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7. $\qquad$ Return $\hat{f}_n \leftarrow \text{Learn}(\mathbb{L})$

Theorem: Agnostic ($\beta = R(f^*)$)
and suppose $f^* =$ global best:
labels

$$\approx d\frac{\beta^2}{\epsilon^2} + \mathfrak{s}_{\epsilon/d} \log\left(\frac{1}{\epsilon}\right)$$

Confirms agnostic sample complexity conjecture
but with extra assumption $f^* =$ global opt.

Near-match lower bound: $d\frac{\beta^2}{\epsilon^2} + \mathfrak{s}_\epsilon + d\log\left(\frac{1}{\epsilon}\right)$

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Principles of Active Learning

1. Query in dense regions where $\hat{f}$ could disagree a lot with f^*

2. Query in regions with low noise

Tsybakov Noise

The alg. adapts to heterogeneity in the noise.

Let’s try it with a model that explicitly describes heterogeneous noise:

Tsybakov Noise

Tsybakov Noise

(Tsybakov, 2004; Mammen & Tsybakov 1999)

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Definition: (Tsybakov noise)
$f^\star(x) = \text{sign}(\eta(x))$ and $\exists \alpha \in (0,1)$ s.t. $\forall \tau > 0$,

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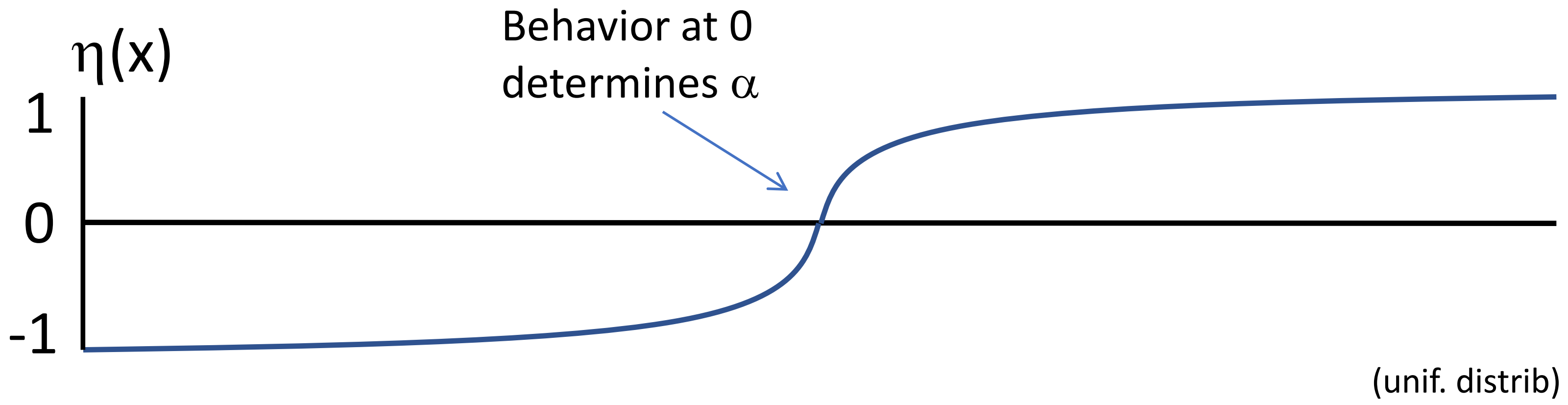
Example:
Thresholds



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Passive OPT: $\tilde{\Theta}\left(\frac{d}{\epsilon^{2-\alpha}}\right)$. (Massart & Nédélec, 2006)

Active OPT: (roughly)

$$\begin{cases} \frac{d}{\epsilon^{2-2\alpha}} & \text{if } 0 < \alpha \leq 1/2 \\ \min\left\{\frac{d}{\epsilon^{2-2\alpha}}\left(\frac{\mathfrak{s}}{d}\right)^{2\alpha-1}, \frac{d}{\epsilon}\right\} & \text{if } 1/2 < \alpha < 1 \end{cases}.$$

(Hanneke & Yang, 2015)

$$\sim \begin{cases} \frac{1}{\varepsilon^{2-2\alpha}}, & \text{if } \mathfrak{s} < \infty \\ \frac{1}{\varepsilon}, & \text{if } \mathfrak{s} = \infty \end{cases}.$$

Active Opt $\ll$ Passive Opt.
(always)

Conclusions

- Many proposals for going beyond Disagreement-based Active Learning
- Each exhibits improvements in certain cases
- We still don't know the **optimal agnostic active learning algorithm**

$$d\frac{\beta^2}{\epsilon^2} + \mathfrak{s}_{\epsilon/d}\log(\frac{1}{\epsilon})$$

Questions?

Further reading:

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